SOME PROBLEMS IN BLOCKING SETS

By

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In this dissertation some new results are obtained on the cardinality of blocking sets in block designs. Firstly, lower bounds are established on the cardinality of blocking sets of Rédei type in finite projective planes. Secondly, a block design Σ is formed using the Hermitian unitals of PG(2,q^2), q an odd prime power, as blocks, and then a lower and upper bound are established on the cardinality of a committee of Σ; also, a characterization of the Desarguesian projective plane PG(2,9) is established.

CHAPTER 1

INTRODUCTION

An *incidence structure* is an ordered pair (A, B), where A is a set and B a collection of subsets of A. The elements of A are called *points* and those of B are called *blocks*. A $t-(v,k,\lambda)$ *design* is an incidence structure (A, B) where the cardinality of A is v, each block of B contains exactly k points and each subset of t points of A is contained in exactly λ blocks of B. For $t = 2$ one calls such an incidence structure a *block design*. A *projective plane* of order n, $n \geq 2$, is a block design with $v = n^2 + n + 1$, $k = n + 1$ and $\lambda = 1$. If Π represents a projective plane of order n, then it is straightforward to show that each pair of distinct blocks of Π intersect in a unique point, and that there are four distinct points of Π no three of which are contained in the same block. A block of a projective plane is called a *line*.

A *hitting set* H of an incidence structure $I = (A, B)$ is a subset of A so that each block of B has non-empty intersection with H. If H is a hitting set of I so that no block is a subset of H, then H is called a *blocking set* of I.

The purpose of this dissertation is to prove some original theorems concerning the cardinality of blocking sets of block designs.

CHAPTER 2

A BOUND FOR BLOCKING SETS OF REDEI TYPE, PART I

We begin Chapter 2 by introducing the notation and terminology that will be used throughout Chapters 2, 3, 4 and 5. Most of the lemmas proven in Chapter 2 will also find use in the succeeding three chapters.

Let Π represent a finite projective plane of order n, and let $\mathcal{S}$ represent a blocking set of Π. For each line l of Π, call the number $\mid \mathcal{S} \cap l \mid$ the *strength of* l and denote it by $st(l)$. If st(l)=$i \geq 2$, then call l a *secant* or an $i-$ *line*. If st(l)=1, then call l a *tangent*.

LEMMA 2.1. *There exists an integer* $\delta \geq 1$ *such that* $\mid \mathcal{S} \mid = n + \delta$.

PROOF: Let l be a line of Π. One can choose a point X on l yet not in $\mathcal{S}$. The n lines other than l through X are each incident with at least one point of $\mathcal{S}$. Thus, $\mid \mathcal{S} \mid \geq n$+st($l$).

Q.E.D.

LEMMA 2.2. *Let* $\mathcal{S}$ *be such that* $\mid \mathcal{S} \mid = n + \delta$. *Let* g *and* h *represent distinct lines of* Π. *If st(g)+st(h)*$\geq \delta + 2$, *then the point* $g \cap h$ *is in* $\mathcal{S}$.

PROOF: Assume, by way of contradiction, that for distinct lines g and h one has st(g)+st(h)$\geq \delta+2$ and $(g \cap h) \notin \mathcal{S}$. The $n-1$ lines other than g and h through point $g \cap h$ are each incident with at least one point of $\mathcal{S}$. Thus, $\mid \mathcal{S} \mid \geq n-1$+st($g$)+st($h$) $\geq n + \delta + 1$, a contradiction.

Q.E.D.

LEMMA 2.3. *Let $\mathcal{S}$ be such that $\mid \mathcal{S} \mid = n+\delta$. Then for any line l one has st(l)$\leq \delta$.*

PROOF: Assume, by way of contradiction, that st(l)$> \delta$ for some secant l. Choose point $X \in l \setminus \mathcal{S}$. The n lines other than l through X are each incident with at least one point of $\mathcal{S}$. Thus, $\mid \mathcal{S} \mid \geq n$+st($l$) $> n+\delta$, a contradiction.

Q.E.D.

One says that a blocking set $\mathcal{S}$ is of *Rédei type* if $\mid \mathcal{S} \mid = n+\delta$ implies the existence of a secant l of strength δ. Such a secant l will sometimes be called a secant of *maximum strength*. By Lemma 2.2, if $\mathcal{S}$ is of Rédei type, then every secant intersects every secant of maximum strength in a point of $\mathcal{S}$.

Let $\mathcal{S}$ be of Rédei type with $\mid \mathcal{S} \mid = n+\delta$. Let U represent a point of Π. One says that U is of *type* $\alpha = (\alpha_1, ..., \alpha_m)$ if U lies on exactly m secants $l_1, ..., l_m$ with α_i=st(l_i) and $\alpha_1 \leq ... \leq \alpha_m$. For point U of type α, define the constant $F(U)$=$\alpha_1 + ... + \alpha_m$; arguments using the computation of F(U) will often be called a *fan count on U*. For point U and integer $i \geq 2$, let $r_i(U)$, or just r_i if the point U is clear, equal the number of i-lines through U, and define $r(U) = r_2 + ... + r_\delta$. Also, for integer $i \geq 2$, let b_i equal the total number of i-lines and b the total number of secants.

LEMMA 2.4. *Let $\mathcal{S}$ be a blocking set of Rédei type with $\mid \mathcal{S} \mid = n+\delta$. Let $P \in \mathcal{S}$ be a point not incident with every secant of maximum strength. Then $F(P) = n+2\delta-1$.*

PROOF: Let l be a δ-line not incident with point $P \in \mathcal{S}$. Then $F(P) = \mid \mathcal{S} \mid + (\delta - 1)$ since every secant through P meets l at a point of $\mathcal{S}$.

Q.E.D.

LEMMA 2.5. *Let $\mathcal{S}$ be a blocking set of Rédei type with $\mid \mathcal{S} \mid = n+\delta$. Let $X \notin \mathcal{S}$ be a point of type $\alpha = (\alpha_1, ..., \alpha_m)$. Then $F(X) = \delta - 1 + m$.*

PROOF: This should be clear since $| \mathcal{S} |= F(X) + (n + 1 - m)$.

Q.E.D.

LEMMA 2.6. *Let $\mathcal{S}$ be a blocking set of Rédei type with $| \mathcal{S} |= n + \delta$. Assume that there is a unique secant l of maximum strength. Let $j \geq 3$ be some fixed integer. If every $P \in \mathcal{S} \setminus l$ lies on a secant g with st(g)$\geq j$, then every point $P \in \mathcal{S} \setminus l$ lies on a secant h with st(h)$\leq \delta + 1 - j$.*

PROOF: Assume, by way of contradiction, that every point in $\mathcal{S} \setminus l$ lies on a secant g with st$(g) \geq j$ and point $P \in \mathcal{S} \setminus l$ is of type α with $\alpha_1 \geq \delta + 2 - j$. Since $\delta + 2 - j \geq 3$, one can choose a point $Q \in \mathcal{S} \setminus l$ such that $PQ = l_1$. By assumption, Q lies on a secant g with st$(g) \geq j$. Since $\alpha_1 \geq \delta + 2 - j$, Lemma 2.2 implies that secant g intersects the δ secants through P at points of $\mathcal{S}$. Thus, st$(g) \geq \delta$, a contradiction since $Q \notin l$, the unique δ-line.

Q.E.D.

LEMMA 2.7. *Let $\mathcal{S}$ be a blocking set with $| \mathcal{S} |= n + \delta$. Let P, Q and R be three distinct points of $\mathcal{S}$ such that $PQ \neq PR$ and st(PR)=j. Then through Q there are at most $j - 1$ secants $g \neq PQ$ of strength $\delta + 2 - j$ or more.*

PROOF: By Lemma 2.2, a secant $g \neq PQ$ through Q of strength $\delta + 2 - j$ or more intersects line PR at a point of $\mathcal{S}$. There are only $j - 1$ points of PR where g can intersect.

Q.E.D.

For secant g, define g^* to be the set of points in $g \setminus \mathcal{S}$, and define L(g^*) to be the set of secants $h \neq g$ such that the point $g \cap h$ is not in $\mathcal{S}$.

LEMMA 2.8. *Let $\mathcal{S}$ be a blocking set with $|\mathcal{S}| = n+\delta$. Let g be a secant of strength less than δ. Then*

$$n+1-st(g) = |g^*| \leq |L(g^*)| .$$

PROOF: This should be clear since Lemma 2.5 and the fact $\mathrm{st}(g) < \delta$ imply that each $X \in g \setminus \mathcal{S}$ lies on at least two secants.

Q.E.D.

We are now ready to introduce the main objective of Chapters 2 and 3.

It was proven by A. Bruen that for any blocking set $\mathcal{S}$, $|\mathcal{S}| \geq n+n^{1/2}+1$. [8, Theorem, and; 9, Theorem 3.8] Under the additional assumption that n is a non-square greater than 5, J. Bierbrauer proved that $|\mathcal{S}| > n+n^{1/2}+2$. [1, Theorem] (If n is square, then Baer subplanes offer counterexamples.) If $n = 10$ and $\mathcal{S}$ is of Rédei type, then J. Bierbrauer showed that $|\mathcal{S}| > 10+10^{1/2}+3$. [3, Theorem] The main objective of Chapters 2 and 3 is to extend this last result and prove the following theorem.

THEOREM 2.9. *Let Π be a finite projective plane of non-square order $n = t^2 - e$, $t \geq 4$, $1 \leq e \leq 2t-2$ and $n \neq 10$. Let $\mathcal{S}$ be a blocking set of Rédei type. Then $|\mathcal{S}| \geq n+t+3$.*

An immediate consequence of Theorem 2.9 and an observation by Bierbrauer [1, Corollary] is the following.

COROLLARY 2.10. *A net of non-square order $n \geq 11$ with more than $n-(n^{1/2}+2)$ parallel classes can be completed in at most one way to a projective plane of order n.*

In view of J. Bierbrauer's work in [1, Theorem], Theorem 2.9 can be proven by showing that $|\mathcal{S}| \neq (t^2-e)+t+2$. Henceforth, assume $|\mathcal{S}| = (t^2-e)+t+2$.

CLAIM 2.11. *([2, Theorem]) There is exactly one $(t+2)$-line.*

Henceforth, let l denote the unique secant of strength $t+2$.

CLAIM 2.12. *For every point $P \in \mathcal{S} \setminus l$,*

$$t^2 + 5 \leq F(P) \leq t^2 + 2t + 2.$$

PROOF: This follows immediatedly from Lemma 2.4.

Q.E.D.

CLAIM 2.13. *Every point $P \in \mathcal{S} \setminus l$ lies on a secant of strength $t-1$ or more.*

PROOF: If not, then

$$t^2 + 5 \leq F(P) \leq (t+2)(t-2) = t^2 - 4.$$

Q.E.D.

By Lemma 2.6, every point $P \in \mathcal{S} \setminus l$ lies on a secant of strength 4 or less.

CLAIM 2.14. *Every point $P \in \mathcal{S} \setminus l$ lies on a secant of strength t or $t+1$.*

PROOF: Assume not and let P be of type α with $\alpha_{t+2} = t-1$. Since $\alpha_1 \leq 4$, one has

$$t^2 + 5 \leq F(P) \leq (t+1)(t-1) + 4 = t^2 + 3.$$

Q.E.D.

By Lemma 2.6, every point $P \in \mathcal{S} \setminus l$ lies on a secant of strength 2 or 3.

CLAIM 2.15. *Let g be a secant of strength $t+1$. Every $P \in \mathcal{S} \setminus (l \cup g)$ lies on exactly one 2-line in $L(g^*)$.*

PROOF: Let $P \in \mathcal{S} \setminus (l \cup g)$. Set $A = l \cap g$ and let $Q_1, \ldots, Q_t$ be the t points in $(g \cap \mathcal{S}) \setminus \{A\}$. The t distinct secants PQ_i intersect l at t distinct points of $\mathcal{S} \setminus \{A\}$.

There is thus one point $B \neq A$ on l and in $\mathcal{S}$ such that $B \notin PQ_i$ for $1 \leq i \leq t$. The secant $h = PB$ intersects g at a point not in $\mathcal{S}$. By Lemma 2.2, h must be a 2-line.

Q.E.D.

If there is a secant g of strength $t+1$, then Claim 2.15 implies that the total number of secants $b = r(A) + t(t+1) + (n-t)$, where A denotes the point $l \cap g$.

CLAIM 2.16. *No point $A \in \mathcal{S} \cap l$ lies on n secants of strength 2.*

PROOF: If A lies on n secants of strength 2, then the set $\mathcal{S} \setminus \{A\}$ is a blocking set with cardinality $n+t+1$, a contradiction of the Bierbrauer bound in [1, Theorem].

Q.E.D.

For the remainder of Chapter 2, assume that $n \geq 37$. The arguments used in dealing with non-square $11 \leq n \leq 35$ are of an ad hoc nature and will be handled in Chapter 3. Note that $n \geq 37$ implies that $t \geq 7$.

CLAIM 2.17. *For every $P \in \mathcal{S} \setminus l$ of type α, one has*

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \geq 14.$$

PROOF: If not, then

$$t^2 + 5 \leq F(P) \leq (t-2)(t+1) + 13 = t^2 - t + 11,$$

a contradiction for $t \geq 7$.

Q.E.D.

CLAIM 2.18. *Every point $P \in \mathcal{S} \setminus l$ lies on a 2-line.*

PROOF: Assume that some P of type α does not lie on a 2-line. Then by Claim 2.14 and Lemma 2.6 $\alpha_1 = 3$ and all secants of strength $t+1$ pass through P. Choose

$Q \in \mathcal{S} \setminus l$ so that $PQ = l_1$. Claim 2.14 and $\alpha_1 = 3$ imply that Q lies on a t-line. It follows from Lemma 2.2 and Claim 2.17 that P lies on exactly three 3-lines. A fan count on P now shows that there are $(t+1)$-lines through P and, hence, Claim 2.15 implies that Q lies on a 2-line. A fan count on Q shows that Q is of type $(2, 3, t, ..., t)$. Let $R \in \mathcal{S} \setminus l$ be such that $PR = l_2$. The secant $g = QR$ must be of strength t. But, $\alpha_4 \geq 4$ implies that g intersects at least $t+1$ secants through P at points of $\mathcal{S}$, a contradiction.

Q.E.D.

Claim 2.19. *None of the points $P \in \mathcal{S} \setminus l$ lies on a 3-line.*

Proof: Assume that some P does lie on a 3-line. Let P be of type α. Claim 2.17 implies that P lies on at most three secants of strength 3 or less. There are three cases to consider.

Case 1. P lies on two 2-lines and one 3-line.

Let $Q \in \mathcal{S} \setminus l$ be such that st(PQ)=3. A fan count on P shows that $\alpha_4 \geq t$ and $\alpha_5 = t + 1$. Since $t + 2 \geq 9$, there are at least five $(t + 1)$-lines through P. Thus, PQ is the only 3-line of $\mathcal{S}$. Since P lies on two 2-lines, Q lies on at most two $(t + 1)$-lines. Since $\alpha_4 \geq t$, Q lies only on secants of strength 2, 3, t or $t + 1$. A fan count on Q and Claim 2.18 imply that Q lies on a 2-line, a 3-line and t secants of strength t or $t+1$. Let QB be the 2-line, and remember that PQ is the only 3-line of $\mathcal{S}$. Then every i-line g through B with $i > 3$ must meet the $t+2$ secants through Q at points of $\mathcal{S}$. This contradiction of Claim 2.11 implies that B must lie on n secants of strength 2, a contradiction to Claim 2.16.

Case 2. P lies on one 2-line and two 3-lines.

Let $Q \in \mathcal{S} \setminus l$ of type β and $B \in \mathcal{S} \cap l$ be such that PQ and PB are a 3-line and a 2-line, respectively. A fan count of P shows that $\alpha_4 \geq t-1$ and $\alpha_6 = t+1$. Since $t+2-5 = t-3 > 3$ for $t \geq 7$, all 3-lines pass through P. Let $g \neq PQ$ be a secant through Q with $\mathrm{st}(g) > 2$. Since $\alpha_6 = t+1$, secant g has strength at least $t-2$. Since $\alpha_4 + (t-2) \geq t+4$ for $t \geq 7$, Lemma 2.2 implies that g has strength at least t. Thus, Q lies on one 2-line, one 3-line and t secants of strength at least t. It also follows that st(QB)=2. Let $h \neq l$ be a secant through B with $\mathrm{st}(h) > 2$. Since $\beta_3 \geq t$, $\beta_2 > 2$ and the fact that all 3-lines pass through P, secant h intersects the $t+2$ secants through Q at points of $\mathcal{S}$, a contradiction. Hence, B lies on n secants of strength 2, a contradiction of Claim 2.16.

Case 3. P lies on a 2-line, a 3-line and $\alpha_3 \geq 4$.

Let $Q \in \mathcal{S} \setminus l$ and $B \in \mathcal{S} \cap l$ be such that PQ=l_2 and PB=l_1, respectively. Set $\beta = \alpha_3$. As Case 1 and Case 2 do not occur, Q also lies on exactly one 2-line and exactly one 3-line.

Subclaim 2.20. If $\beta \leq t-1$, then Q lies on β secants of strength $t+4-\beta$ or larger.

Proof. Assume not. Then,

$$t^2 + 5 \leq F(Q) \leq (\beta - 1)(t+1) + (t - \beta + 1)(t + 3 - \beta) + 5.$$

Equivalently,

$$f(\beta) := \beta^2 - \beta(t+3) + 3t + 2 \geq 0;$$

but,

$$f(\beta) < 0 \text{ for } 4 \leq \beta \leq t-1,$$

a contradiction.

q.e.d.

If $\beta \leq t-1$, then Subclaim 2.20 would contradict Lemma 2.7. Thus, $\beta \geq t$, or P lies on t secants of strengths t or $t+1$. The same conclusion applies to Q, and Lemma 2.7 then yields $\beta = t+1$. It is then immediate that B must lie on n secants of strength 2, a contradiction of Claim 2.16.

Q.E.D.

CLAIM 2.21. *None of the points $P \in \mathcal{S} \setminus l$ lies on a 4-line.*

PROOF: Assume some P of type α does lie on a 4-line. Since α_1=2, a fan count on P shows that P lies on at most three 4-lines. We consider four cases.

Case 1. P lies on two 4-lines and $\alpha_4 \geq 5$.

Subclaim 2.22. For $1 \leq k \leq t-5$,

$$r_{t+1-k} + \ldots + r_{t+1} \geq k+3.$$

Proof. Assume not. Then,

$$t^2 + 5 \leq F(P) \leq (k+2)(t+1) + (t-k-3)(t-k) + 4 + 4 + 2.$$

Equivalently,

$$f(k) := k^2 + k(4-t) - t + 7 \geq 0;$$

but, $f(k) < 0$ for $1 \leq k \leq t-5$, a contradiction.

q.e.d.

Let $B \in \mathcal{S} \cap l$ be such that PB=l_1, and let $g \neq l$ be an i-line through B with $i > 2$. By Claim 2.19, $i \geq 4$, so g intersects secants of strength t or $t+1$ at points of $\mathcal{S}$. As g also contains B, Subclaim 2.22 implies that $i \geq (1+3)+1 = 5$. Using this argument repeatedly, one gets that $i \geq (t-5+3)+1 = t-1$. As $\alpha_4 \geq 5$, Lemma 2.2 implies that $i \geq t$. Thus, g intersects the $t+2$ secants through P at points of $\mathcal{S}$, a contradiction to Claim 2.11. Hence, there must be n secants of strength 2 through B, a contradiction to Claim 2.16.

Case 2. P lies on two 2-lines and a 4-line.

Let $Q \in \mathcal{S}\setminus l$ be such that $PQ = l_3$. A fan count on P shows that $F(P) \leq t^2+7$, $\alpha_4 \geq t-1$ and $\alpha_5 \geq t$. Since P lies on two 2-lines, Q lies on at most two $(t+1)$-lines. A fan count on Q reveals that Q lies on a 2-line, a 4-line and t secants of strength $t-3$ or more. Using the bounds on α_4 and α_5, one sees that Q lies on t secants of strength t or $t+1$. Let $B \in \mathcal{S} \cap l$ be such that QB is a 2-line. Remembering that there are no 3-lines, it is immediate that B must lie on n secants of strength 2, a contradiction to Claim 2.16.

Case 3. P lies on three 4-lines.

Let $Q \in \mathcal{S} \setminus l$ and $B \in \mathcal{S} \cap l$ be such that $PQ = l_2$ and $PB = l_1$, respectively. A fan count on P shows that $t = 7$ and $\alpha_5 = 8$. This implies that any secant $g \neq l$ through B has strength 2 or 6, and that QB is also a 2-line. Since $\alpha_3 = 4$ and $\alpha_5 = 8$, point Q lies on at most three 8-lines and only on secants of strength 2, 4, 6 or 8. Thus, Q is of type (2,4,6,6,6,6,8,8,8). Lemma 2.2 then implies that B is incident with n 2-lines, a contradiction to Claim 2.16.

Case 4. $\alpha_3 \geq 5$.

Let $Q \in \mathcal{S}\setminus l$ be such that $PQ = l_2$. A fan count on Q shows that if $\alpha_3 \leq t-2$, then Q lies on α_3 secants of strength $t+4-\alpha_3$ or larger. Lemma 2.7 then yields the contradiction $\alpha_3 \geq t-1$. Because Cases 1, 2 and 3 do not occur, Q lies on exactly one 2-line and exactly one 4-line. Since $\alpha_3 \geq t-1$, the remaining t secants through Q are of strength $t+1$. If $B \in \mathcal{S} \cap l$ is such that QB is the 2-line through Q, it is straightforward to show that B must lie on n secants of strength 2, a contradiction to Claim 2.16.

Q.E.D.

CLAIM 2.23. *Every $P \in \mathcal{S} \setminus l$ lies on at least two 2-lines.*

PROOF: Suppose P is of type α and lies on exactly one 2-line. Let $B \in \mathcal{S} \cap l$ be such that $PB = l_1$. First assume either that $t \geq 8$, or that $t = 7$ and $F(P) \geq t^2+6$. Arguing as in Claim 2.21, Case 1, it can be proved that for $2 \leq k \leq t-5$, one has

$$r_{t+1-k} + \dots + r_{t+1} \geq k+3,$$

and, hence, that B lies on n secants of strength 2. This contradiction yields $t = 7$ and $F(P) = t^2+5$. Consider any i-line $g \neq l$ through B with $i > 2$. Claims 2.19 and 2.21 imply that $i \geq 5$. If $\alpha_5 \geq 6$, then P lies on five secants of strength 6 or more. This fact and Lemma 2.2 imply that $i \geq 6$. But, as $\alpha_2 \geq 5$, Lemma 2.2 implies that $i = t+2$. This contradiction implies that $\alpha_5 = 5$, and, hence, that P is of type (2,5,5,5,5,8,8,8,8). This implies that there are no 6-lines or 7-lines. Thus, any point $Q \in \mathcal{S} \setminus l$ is either of type (2,5,5,5,5,8,8,8,8), (2,2,5,5,8,8,8,8,8) or (2,2,2,8,8,8,8,8,8). Assume that Q is of one of the latter two types, and let $R \in \mathcal{S} \setminus l$ be any point other than Q. Since Q lies on two 2-lines, R lies on at most three 8-lines. This contradiction yields that all points P are of type (2,5,5,5,5,8,8,8,8). Counting flags (T,g), where $T \in \mathcal{S} \setminus l$ and g is an 8-line, produces the contradiction $7b_8 = 148$.

Q.E.D.

CLAIM 2.24. *None of the points $P \in \mathcal{S} \setminus l$ lies on three 2-lines.*

PROOF: Suppose P lies on three 2-lines. A fan count on P shows that P lies on a $(t+1)$-line. Let $Q \in \mathcal{S} \setminus l$ be such that PQ is a $(t+1)$-line. Since P lies on three 2-lines, Q lies on exactly one $(t+1)$-line and at most three t-lines. Claim 2.23 yields

$$t^2+5 \leq \mathrm{F(P)} \leq (t+1)+3t+(t-1)+4 = 5t+4,$$

or, $t \leq 4$, a contradiction.

Q.E.D.

Proof of Theorem 2.9 for $n \geq 37$. By Claims 2.23 and 2.24, every point $P \in \mathcal{S} \setminus l$ lies on exactly two 2-lines. Thus, no point $P \in \mathcal{S} \setminus l$ lies on more than three $(t+1)$-lines. If P is of type α, then a fan count on P reveals that P lies on a t-line and $\alpha_4 \geq t-1$. Let $Q \in \mathcal{S} \setminus l$ be of type β such that PQ is a t-line. Since Q lies on two 2-lines, P lies on at most two $(t+1)$-lines. Since $\beta_4 \geq t-1$ and $\beta_3 \geq 5$, P lies on t secants of strength t or $t+1$. A fan count on P now shows that $t^2+5 \leq F(P) \leq t^2+6$. If $F(P) = t^2+5$, then all points P are of type $(2, 2, t, ..., t, t+1)$. If $F(P) = t^2+6$, then all points P are of type $(2, 2, t, ..., t, (t+1), (t+1))$. In either case, a flag count of (T, g), where $T \in \mathcal{S} \setminus l$ and g is a $(t+1)$-line, reveals a contradiction.

CHAPTER 3

A BOUND FOR BLOCKING SETS OF REDEI TYPE, PART II

As previously stated, the objective of Chapter 3 is to prove Theorem 2.9 for $11 \leq n \leq 35$, n non-square. Note that this implies $4 \leq t \leq 6$, and; if $P \in \mathcal{S} \setminus l$, then $\mathrm{F}(P) \geq t^2+5$ for $t=5$ or 6 and $\mathrm{F}(P) \geq t^2+6$ for $t=4$. Again, the terminology and notation used in Chapter 2 will also be employed here. Let us also add that P, Q, R, T and U will always represent points in $\mathcal{S} \setminus l$, and A and B will always represent points in $\mathcal{S} \cap l$.

Claim 3.1. *Let P be of type α with $\alpha_{t+2}=t$. Then α_1=2.*

Proof: Assume, by way of contradiction, that P is of type α with $\alpha_{t+2}=t$ and $\alpha_1 \neq 2$. By Lemma 2.6 and Claim 2.14, $\alpha_1=3$. Let Q be such that $PQ=l_1$. By Claim 2.14, there is a secant g through Q such that st(g)=t or $t+1$. Since

$$(t-1)t+3\times 3=t^2-t+9<t^2+5$$

for t=5 or 6, and since

$$t^2-t+9<t^2+6$$

for t=4, one has $\alpha_3 \geq 4$. Hence, Lemma 2.2 implies that st(g)=t+2, a contradiction.

Q.E.D.

Claim 3.1, Claim 2.14 and Lemma 2.6 imply that every P lies on a 2-line.

Claim 3.2. *Let P be of type α with $\alpha_{t+2} = t$. Set $g = l_{t+2}$. Then every Q not on g lies on exactly two secants from $L(g^*)$.*

Proof: This should be clear; see Claim 2.15.

Q.E.D.

Claim 3.3. *Let P be of type α with $\alpha_{t+2} = t$. Then $\alpha_{t+3-i} \geq t + 4 - i$ for $4 \leq i \leq t+1$.*

Proof: Assume, by way of contradiction, that $\alpha_{t+3-i} < t+4-i$ for some i. Then

$$F(P) \leq (i-1)t + (t+2-i)(t+3-i) + 2 = i^2 - i(t+5) + t^2 + 4t + 8.$$

If $4 \leq i \leq t+1$, then the inequality yields the contradiction $\mathrm{F}(P) < t^2 + 5$.

Q.E.D.

Claim 3.4. *Let P be of type α with $\alpha_{t+2} = t$. Then $\alpha_2 \neq 3$.*

Proof: Assume, by way of contradiction, that there is a P of type α with $\alpha_{t+2} = t$ and $\alpha_2 = 3$. A computation of $\mathrm{F}(P)$ reveals that P is of type $(2,3,t,...,t)$ and, hence, $\mathrm{F}(P)= t^2 + 5$. Thus, t=5 or 6. Choose Q of type β such that PQ=l_2. By Claim 3.3, Q lies only on secants of strength 2, 3 or $t+1$. By Claim 3.1, $\beta_1 = 2$. By Claim 3.2, $\beta_3 \leq 3$. Since

$$(t-2)(t+1) + 3 + 8 = t^2 - t + 9 < t^2 + 5$$

for $t \geq 5$, Q lies on $t-1$ secants of strength $t+1$. Thus,

$$F(Q) \geq (t-1)(t+1) + 7 = t^2 + 6,$$

a contradiction.

Q.E.D.

CLAIM 3.5. *Every P lies on a $(t+1)$-line.*

PROOF: Assume, by way of contradiction, that P is of type α with $\alpha_{t+2} = t$. By Claim 3.1, $\alpha_1 = 2$. A computation of F(P) shows that $\alpha_2 \geq 3$. Claim 3.4 then implies $\alpha_2 \geq 4$. Choose distinct points Q and R on l_2 not equal to P. By Claim 3.3, Q lies only on secants $g \neq l_2$ of strength 2, 3 or $t+1$. Let A be such that $PA = l_1$. Claim 3.3 and the fact $\alpha_2 \geq 4$ imply that A lies only on secants $g \neq l$ of strength 2 or 3.

Let $t = 4$. Then P is of type (2,4,4,4,4,4) and F(P)=22. Point Q lies on exactly one 4-line. Thus, as is easily seen, all points $T \neq P$ are of type (2,3,3,4,5,5). If QA is a 3-line, then the two 5-lines through R can not intersect QA at points of $\mathcal{S}$. Thus, A lies on eleven 2-lines, a contradiction of Claim 2.16. Thus, $t \neq 4$.

Since

$$2(t+1) + 1 \times 2 + \alpha_2 + (t-2)3 = 5t - 2 + \alpha_2 < t^2 + 5$$

for $t = 5$ or 6, Q and R each lie on at least three $(t+1)$-lines. If A lies on a secant g of strength 3, then g passes through Q and R, a contradiction. Thus, A lies on n secants of strength 2, a contradiction of Claim 2.16.

Q.E.D.

CLAIM 3.6. *Let P be of type α. Then $\alpha_2 \neq 6$.*

PROOF: Assume, by way of contradiction, that P is of type α with $\alpha_2 = 6$. Since

$$(t+1)6 + 2 = 6t + 8 > t^2 + 2t + 2$$

for $t = 4$ or 5, $t = 6$ and F$(P)\geq 45$. Let Q and R be distinct points on l_2 not equal to P. Since $\alpha_2 = 6$, Q and R lie only on secants $g \neq l_2$ of strength 2, 3 or 7. Since

$$4 \times 7 + 1 \times 6 + 1 \times 2 + 2 \times 3 = 42 < 45,$$

Q and R each lie on at least five 7-lines. Let A be such that $PA=l_1$. As $\alpha_2 = 6$, A lies only on secants $g \neq l$ of strength 2 or 3. If A lies on a 3-line h, then either all the 7-lines through Q or all the 7-lines through R can not intersect with h at points of $\mathcal{S}$, a contradiction. Thus, A lies on n secants of strength 2, a contradiction of Claim 2.16.

Q.E.D.

CLAIM 3.7. *Let P be of type α. Then $\alpha_2 \neq 5$.*

PROOF: Assume, by way of contradiction, that P is of type α with $\alpha_2 = 5$. Since

$$F(P) \geq (t+1) + t \times 5 + 2 = 6t + 3 > t^2 + 2t + 2$$

for $t = 4$, one has $t = 5$ or 6. Let Q and A be such that $PQ = l_2$ and $PA = l_1$.

Suppose $t = 5$. Then Q lies only on secants $g \neq l_2$ of strength 2, 3 or 6. Since

$$2 \times 6 + 1 \times 5 + 1 \times 2 + 3 \times 3 = 28 < 30,$$

Q lies on at least three 6-lines. Point A lies only on secants $g \neq l$ with st(g)=2 or 3. If A lies on a 3-line h, then the 6-lines through Q can not all intersect h at points of $\mathcal{S}$, a contradiction, unless $QA = h$. As $\alpha_2 > 3$, there is a point $R \in l_2$ such that the 6-lines through R do not all intersect h at points of $\mathcal{S}$. Thus, A lies on n secants of strength 2, a contradiction of Claim 2.16. So, $t = 6$.

Point Q lies only on secants $g \neq l_2$ with st(g)=2, 3, 4 or 7. Since

$$3 \times 7 + 1 \times 5 + 1 \times 2 + 3 \times 4 = 40 < 41,$$

Q lies on at least four 7-lines. Point A lies only on secants $g \neq l$ with st(g)=2, 3 or 4. An argument similar to the one for $t = 5$ will show that A lies on n secants of strength 2, a contradiction.

Q.E.D.

CLAIM 3.8. *Let $n \neq 17$ and P be of type α. Then $\alpha_2 \neq 4$.*

PROOF: Assume, by way of contradiction, that $n \neq 17$ and P is of type α with $\alpha_2 = 4$. Let Q and R be distinct points on l_2 not equal to P. Let A be such that $PA = l_1$.

Suppose $t = 4$. Then F(P)≥ 23, Q and R lie on exactly one 4-line and A lies only on secants $g \neq l$ of strength 2 or 3. Since

$$2 \times 5 + 1 \times 4 + 1 \times 2 + 2 \times 3 = 22 < 23,$$

Q and R lie on at least three 5-lines. Let h be a 3-line through A. Secant h must intersect each 5-line through Q or R at a point of $\mathcal{S}$. Hence, h passes through Q and R, a contradiction. Thus, A lies on n secants of strength 2, a contradiction of Claim 2.16. So, $t = 5$ or 6.

Suppose $\alpha_4 \geq 5$. Then neither Q nor R lie on a secant $g \neq l_2$ with st(g)$= t-1$ or t , and A does not lie on a secant of strength $t-1, t$ or $t+1$. Since

$$(t-3)(t+1) + 1 \times 4 + 1 \times 2 + 3(t-2) = t^2 + t - 3 < t^2 + 5$$

for $t = 5$ or 6, points Q and R both lie on at least $t-2$ secants of strength $t+1$. Any secant h through A such that $3 \leq st(h) \leq t-2$ intersects each of the $(t+1)$-lines through Q or R at points of $\mathcal{S}$. Thus, h passes through Q and R, a contradiction. Hence, A lies on n secants of strength 2, a contradiction. So, $\alpha_4 = 4$.

Suppose $t = 5$. Remember that $n \neq 17$. Since $\alpha_4 = 4$, F(P)=31 or 32 and point Q lies on at most three 6-lines and no 5-lines. As

$$2 \times 6 + 1 \times 4 + 1 \times 2 + 3 \times 4 = 30 < 31,$$

Q lies on exactly three 6-lines and, hence, does not lie on a 3-line. Thus, Q and P are both of type $(2,4,4,4,6,6,6)$ and F(P)=32. Let T be such that st(PT)=6. T lies on at most four 6-lines and no 5-lines. A computation of F(T) reveals that T lies on at least three 6-lines. If T lies on four 6-lines, then T is of type $(2,3,3,6,6,6,6)$ and some point in $\mathcal{S} \setminus l$ lies on at most two 6-lines, a contradiction. Thus, all points in $\mathcal{S} \setminus l$ are of type $(2,4,4,4,6,6,6)$. Counting flags (U,g), $U \in \mathcal{S} \setminus l$ and g a 6-line, yields the contradiction

$$19 \times 3 = 5b_6.$$

So, $t = 6$.

Since $\alpha_4 = 4$, P lies on at least three 7-lines, $\alpha_5 \geq 6$ and F(P)=41 or 42. Thus, Q and R do not lie on secants of strength 3 or 6 and lie on exactly one 4-line. A computation of F(Q) shows that Q and R are both of type (2,4,5,5,5,7,7,7). But, it is not possible that both Q and R lie on distinct 5-lines and six secants of strength 5 or more. So, $t \neq 6$.

Q.E.D.

Claim 3.9. *Let $t = 5$ or 6 and P be of type α with $\alpha_2 = 3$. Then $\alpha_3 \neq 4$.*

Proof: Assume, by way of contradiction, that $t = 5$ or 6 and P is of type α with $\alpha_2 = 3$ and $\alpha_3 = 4$. Let Q be of type β such that $PQ = l_2$. Since $\alpha_3 = 4$, Q does not lie on a t-line and lies on at most three $(t+1)$-lines.

Suppose t=5. Since

$$2 \times 6 + 1 \times 3 + 1 \times 2 + 3 \times 4 = 29 < 30,$$

Q lies on exactly three 6-lines. Since

$$3 \times 6 + 1 \times 3 + 2 \times 2 + 1 \times 4 = 29 < 30,$$

$\beta_2 = 3$. Thus, Q is either of type $(2,3,3,4,6,6,6)$ or $(2,3,4,4,6,6,6)$. Let R be of type γ such that st(QR)=4. Since $\beta_5 = 6$, R does not lie on a 3-line. If $\gamma_2 = 2$, then Q lies on at most two 6-lines. Hence, Claims 3.6, 3.7 and 3.8 imply that $\gamma_2 = 4$ and $n = 17$. So, Q is of type $(2,3,3,4,6,6,6)$. Since $\beta_4 = 4$, P lies on at most three secants of strength 5 or 6 and, hence, is of type $(2,3,4,4,5,6,6)$. Since $\gamma_2 = 4$, all 5-lines pass through R and, hence, R is of type $(2,4,4,4,5,5,6)$. Let T be a point on QR distinct from Q and R. T is of the same type as R. But, this is not possible since R and T both lie on 5-lines and six secants of strength 4 or more. So, $t = 6$.

A computation of F(Q) reveals that Q is of type $(2,3,5,5,5,7,7,7)$. Thus, P does not lie on a secant of strength 5 or 6 and, hence, is of type $(2,3,4,4,7,7,7,7)$. Let A be such that $PA = l_1$. Since P lies on four 7-lines and $\alpha_4 \geq 4$, QA must also be a 2-line. PA and QA being of strength 2 imply that A lies on n secants of strength 2, a contradiction of Claim 2.16.

Q.E.D.

Part 1. Assume $11 \leq n \leq 15$. Let P be of type α. By Claims 3.7 and 3.8, $\alpha_2 \leq 3$ and, hence, F(P)$\leq$ 25. So, $n \neq 15$. By the Bruck-Ryser Theorem [12, Corollary 2.4], one need not consider $n = 14$.

Assume $n = 13$. Every point P is either of type $(2,2,5,5,5,5)$ or $(2,3,4,5,5,5)$. If P is of type $(2,3,4,5,5,5)$ with PQ the 3-line, then Q can not lie on a 4-line, a contradiction. If P is of type $(2,2,5,5,5,5)$ with PQ a 5-line, then Q is on at most three 5-lines, a contradiction.

Assume $n = 12$. Every point P is either of type $(2,2,4,5,5,5)$, $(2,3,3,5,5,5)$ or $(2,3,4,4,5,5)$. Denote by v_1, v_2 and v_3 the number of points of each type, respectively. If P is of type $(2,2,4,5,5,5)$ with PQ the 4-line, then Q is on no 3-lines and at most two 5-lines. None of the point types satisfy these conditions. Thus, $v_1 = 0$. If P is of type $(2,3,3,5,5,5)$ with PQ a 3-line, then Q is of type

$(2,3,4,4,5,5)$ since it is on at most one 3-line. If P is of type $(2,3,4,4,5,5)$ with PQ the 3-line, then Q is of type $(2,3,3,5,5,5)$ since it is on no 4-lines. So, $v_2 \geq 1$ if and only if $v_3 \geq 1$. Now count:

$$12 = v_2 + v_3$$
$$4b_5 = 3v_2 + 2v_3$$
$$3b_4 = 2v_3.$$

The only solutions have either $v_2 = 0$ or $v_3 = 0$, a contradiction.

Assume $n = 11$. Every P is either of type $(2,2,3,5,5,5)$, $(2,2,4,4,5,5)$, $(2,3,3,4,5,5)$ or $(2,3,4,4,4,5)$. Denote by v_1, v_2, v_3 and v_4 the number of points of each type, respectively. If P is of type (2,2,3,5,5,5) with PQ the 3-line, then Q is on exactly one 3-line and at most two 5-lines. Thus, Q is of type $(2,3,4,4,4,5)$. If P is of type $(2,3,4,4,4,5)$ with Q the 3-line, then Q is on no 4-lines. Thus, Q is of type $(2,2,3,5,5,5)$. So, $v_1 = v_4$. Counting flags (R,g), g a 5-line, yields the contradiction:

$$4b_5 = 3v_1 + 2v_2 + 2v_3 + v_4 = 2(v_1 + v_2 + v_3 + v_4) = 22.$$

This completes Part 1.

Part 2. Assume $17 \leq n \leq 24$. Let P be of type α. By Claims 3.6, 3.7 and 3.8, F(P)≤ 35. Thus, $n \leq 22$. By Bruck-Ryser [12, Corollary 2.4], one need not consider $n = 21$ or 22.

Assume $n = 20$. Every P is of type $(2,2,5,6,6,6,6)$ or $(2,3,5,5,6,6,6)$. If P is of the latter type with PA a 2-line, then Lemma 2.2 and Claim 2.11 imply that A lies on twenty 2-lines, a contradiction to Claim 2.16. If P is of type $(2,2,5,6,6,6,6)$ with PQ the 5-line, then Q lies on at most two 6-lines, a contradiction.

Assume $n = 19$. Every P must be one of four types: $(2,2,4,6,6,6,6)$, $(2,3,3,6,6,6,6)$, $(2,2,5,5,6,6,6)$ or $(2,3,5,5,5,6,6)$. If P is of type $(2,2,4,6,6,6,6)$ with PQ the 4-line, then Q is on exactly one 4-line and at most two 6-lines. None of the four point types satisfy these conditions. If P is of type $(2,2,5,5,6,6,6)$ with PQ a 5-line, then Q lies on no 3-lines and at most two 6-lines. None of the remaining three point types satisfy these conditions.

Thus, P is either of type $(2,3,3,6,6,6,6)$ or $(2,3,5,5,5,6,6)$. Let v_1 and v_2 denote the number of points of each type, respectively. If P is of type $(2,3,3,6,6,6,6)$ with $T \neq P$, then T lies on at most one 3-line. Hence, $v_1 \leq 1$. Counting flags (R,g), g a 6-line, one gets

$$5b_6 = 4v_1 + 2v_2 = 2v_1 + 38.$$

Thus, $v_1 = 1$. Now counting flags (R,g), g a 5-line, yields the contradiction $4b_5 = 54$.

Assume $n = 18$. Every point P is one of five point types: $(2,2,3,6,6,6,6)$, $(2,2,4,5,6,6,6)$, $(2,3,3,5,6,6,6)$, $(2,2,5,5,5,6,6)$ or $(2,3,5,5,5,5,6)$.

If P is of type $(2,2,4,5,6,6,6)$ with PQ the 4-line, then Q is on exactly one 4-line and at most two 6-lines. None of the point types satisfy these conditions.

If P is of type $(2,3,3,5,6,6,6)$ with PQ a 3-line, then Q is on exactly one 3-line, at most four lines of strength 5 or 6 and at most two 6-lines. None of the four remaining point types satisfy these conditions.

Let P be of type $(2,2,3,6,6,6,6)$ with PQ the 3-line and PR a 6-line. Q lies on exactly one 3-line and at most two 6-lines; Q must be of type $(2,3,5,5,5,5,6)$. R is on no 3-lines. Thus, $b_3 = 1$. Reconsider Q of type $(2,3,5,5,5,5,6)$ with QA the 2-line. Since there is only one 3-line and A does not lie on it, A lies on eighteen 2-lines, a contradiction of Claim 2.16.

If P is of type $(2,3,5,5,5,5,6)$ with PQ the 3-line, then point Q lies on at most four lines of strength 4 or greater. Neither of the two remaining point types satisfy these conditions.

Hence, all eighteen points P are of type $(2,2,5,5,5,6,6)$. Counting flags (P,g), g a 6-line, reveals the contradiction $5b_6 = 36$.

Assume $n = 17$. Every P is of one of nine types: $(2,2,2,6,6,6,6)$, $(2,2,3,5,6,6,6)$, $(2,2,4,4,6,6,6)$, $(2,3,3,4,6,6,6)$, $(2,2,4,5,5,6,6)$, $(2,3,3,5,5,6,6)$, $(2,4,4,4,4,6,6)$, $(2,2,5,5,5,5,6)$ or $(2,4,4,4,5,5,6)$.

If P is of type $(2,2,3,5,6,6,6)$ with PQ the 3-line, then Q lies on exactly one 3-line and at most two 6-lines. No point type satisfies these conditions.

If P is of type $(2,2,2,6,6,6,6)$ with $Q \neq P$, then Q lies on exactly one 6-line and at most four secants of strength 5 or 6; Q must be of type $(2,4,4,4,5,5,6)$. Let R be such that st(QR)=4. R can not lie on a 5-line and, hence, is neither of type $(2,2,2,6,6,6,6)$ nor $(2,4,4,4,5,5,6)$, a contradiction.

If P is of type $(2,3,3,4,6,6,6)$ with PQ a 3-line, then Q lies on exactly one 3-line. None of the remaining seven point types satisfy this condition.

If P is of type $(2,4,4,4,4,6,6)$ with $Q \neq P$, then Q does not lie on a 5-line; Q is either of type $(2,2,4,4,6,6,6)$ or $(2,4,4,4,4,6,6)$. Denote by v_1 and v_2 the number of points of each type, respectively. Now count:

$$17 = v_1 + v_2$$

$$5b_6 = 3v_1 + 2v_2 = v_1 + 34$$

$$3b_4 = 2v_1 + 4v_2 = 34 + 2v_2.$$

As is easily verified, either $v_1 = 1$ and $v_2 = 16$, or $v_1 = 16$ and $v_2 = 1$. In either case, $v_1 \geq 1$. If Q is of type $(2,2,4,4,6,6,6)$ with QR a 4-line, then R lies on at most two 6-lines. Thus, $v_2 \geq 4$. Or, $v_1 = 1$ and $v_2 = 16$. Thus, $b_6 = 7$. Now, let

Q be a point of type $(2,2,4,4,6,6,6)$ with QB a 4-line. Without loss of generality, assume PQ is a 6-line and PB a 4-line. Let R and T be the points on PB not equal to P. Clearly, B is on no 6-lines. Hence, each of the seven 6-lines intersect PB at P, R or T. Thus, either P, R or T lies on three 6-lines. Thus, $v_1 \geq 2$, a contradiction. Thus, there are no points of type $(2,4,4,4,4,6,6)$.

Let P be of type $(2,4,4,4,5,5,6)$ with PQ a 5-line. Q lies on exactly one 5-line. None of the remaining five point types satisfy this condition.

If P is of type $(2,2,4,4,6,6,6)$ with PQ a 4-line, then Q is on no 3-lines, at most three lines of strength 5 or 6 and at most two 6-lines. None of the remaining four point types satisfy these conditions.

If P is of type $(2,2,5,5,5,5,6)$ with $Q \neq P$, then Q does not lie on a 4-line. Hence, Q is either of type $(2,2,5,5,5,5,6)$ or $(2,3,3,5,5,6,6)$. Denote by v_1 and v_2 the number of points of each type, respectively. Now count:

$$17 = v_1 + v_2$$

$$5b_6 = 17 + v_2$$

$$4b_5 = 2v_1 + 34$$

$$b_4 = 0$$

$$2b_3 = 2v_2$$

$$b_2 = v_1 + 17.$$

As is easily verified, $v_1 = 9$ and $v_2 = 8$. Thus, $b_6 = 5$, $b_5 = 13$, $b_4 = 0$, $b_3 = 8$ and $b_2 = 26$. So, the total number of secants $b = 53$. Let $A \in \mathcal{S} \cap l$ be such that A lies on a 6-line. By Claim 2.15, $r(A) = 11$. As is easily checked, A is either on one 7-line, one 6-line, one 5-line and eight 2-lines, or on one 7-line, one 6-line, three 3-lines and six 2-lines. Since A is on exactly one 6-line and $b_6 = 5$, there are at least thirty 2-lines, a contradiction.

So, P is either of type (2,2,4,5,5,6,6) or (2,3,3,5,5,6,6). Denote by v_1 and v_2 the number of points of each type, respectively. Counting flags (T,g), g a 6-line, yields the final contradiction $5b_6=2v_1+2v_2=34$.

Part 3. Assume $26 \leq n \leq 35$. Let P be of type α. By Claims 3.6, 3.7 and 3.8, F(P)$\leq$ 47. Thus, $n \leq 32$. By Bruck-Ryser[12, Corollary 2.4], there is no need to consider $n = 30$.

Assume $n = 32$. Then P is of type $(2,3,7,7,7,7,7,7)$. Let PA be the 2-line. Then by Lemma 2.2 and Claim 2.11, A must lie on thirty-two 2-lines, a contradiction of Claim 2.16.

Assume $n = 31$. P is either of type $(2,2,7,7,7,7,7,7)$ or $(2,3,6,7,7,7,7,7)$. Let P be of type $(2,3,6,7,7,7,7,7)$ with PA the 2-line. Then A lies on thirty-one 2-lines, a contradiction. Thus, all points P must be of type $(2,2,7,7,7,7,7,7)$. But, if PQ is a 7-line, then Q lies on at most three 7-lines.

Assume $n = 29$. Then P is one of four types: (2,3,5,6,7,7,7,7), (2,3,6,6,6,7,7,7), (2,2,5,7,7,7,7,7) or (2,2,6,6,7,7,7,7). If P is either of the first two types with PA the 2-line, then A lies on twenty-nine 2-lines, a contradiction of Claim 2.16. Thus, P is either of type $(2,2,5,7,7,7,7,7)$ or $(2,2,6,6,7,7,7,7)$. But, if PQ is a 7-line, then Q lies on at most three 7-lines

Assume $n = 28$. Then P is one of seven type: (2,3,5,5,7,7,7,7), (2,3,5,6,6,7,7,7), (2,3,3,7,7,7,7,7), (2,3,6,6,6,6,7,7), (2,2,4,7,7,7,7,7), (2,2,5,6,7,7,7,7) or (2,2,6,6,6,7,7,7). If P is either of the first two types with PA the 2-line, then A lies on twenty-eight 2-lines, a contradiction of Claim 2.16.

If P is of type $(2,2,4,7,7,7,7,7)$ with PQ the 4-line, then Q lies on at most two 7-lines. None of the remaining five point types satisfy these conditions.

If P is either of type $(2,2,5,6,7,7,7,7)$ or $(2,2,6,6,6,7,7,7)$ with PQ a 6-line, then Q lies on at most two 7-lines and no 3-lines. None of the remaining four point types satisfy these conditions.

Thus, P is either of type $(2,3,3,7,7,7,7,7)$ or $(2,3,6,6,6,6,7,7)$. If P is of the former type with PQ a 7-line, then Q can not lie on a 3-line. Thus, all P must be of type $(2,3,6,6,6,6,7,7)$. Counting flags (T,g), g a 7-line, yields the contradiction, $6b_7 = 56$.

Assume $n = 27$. Then P is one of nine types: (2,3,5,5,6,7,7,7), (2,2,3,7,7,7,7,7), (2,2,4,6,7,7,7,7), (2,3,3,6,7,7,7,7), (2,2,5,5,7,7,7,7), (2,2,5,6,6,7,7,7), (2,2,6,6,6,6,7,7), (2,3,5,6,6,6,7,7) or (2,3,6,6,6,6,6,7). If P is the first type with PA the 2-line, then A is on twenty-seven 2-lines, a contradiction.

If P is either of type $(2,2,5,5,7,7,7,7)$ or $(2,2,5,6,6,7,7,7)$ with PQ a 7-line, then Q is on at most three 7-lines, no 5-lines and at most five lines of strength 6 or 7. None of the remaining eight point types satisfy these conditions.

If P is of type $(2,2,3,7,7,7,7,7)$ with PQ a 7-line, then Q is on no 3-lines, at most three 7-lines and at most five lines of strength 6 or 7. None of the remaining six point types satisfy these conditions.

If P is of type $(2,2,4,6,7,7,7,7)$ with PQ a 7-line, then Q is on no 3-lines and at most four lines of strength 6 or 7. None of the remaining five point types satisfy these conditions.

If P is either of types $(2,3,5,6,6,6,7,7)$ or $(2,3,6,6,6,6,7,7)$ with PQ the 3-line, then Q does not lie on a 6-line. None of the remaining four point types satisfy these conditions.

Thus, P is either of type $(2,2,6,6,6,6,7,7)$ or $(2,3,3,6,7,7,7,7)$. If P is of the later type with PQ a 3-line, then Q is on exactly one 3-line. Hence, all points P

are of type $(2,2,6,6,6,6,7,7)$. Counting flags (T,g), g a 6-line, yields the final contradiction, $5b_6 = 108$.

Assume $n = 26$. P is one of twelve types: (2,2,2,7,7,7,7,7), (2,2,3,6,7,7,7,7), (2,2,4,5,7,7,7,7), (2,3,3,5,7,7,7,7), (2,2,4,6,6,7,7,7), (2,3,3,6,6,7,7,7), (2,2,5,5,6,7,7,7), (2,3,5,5,5,7,7,7), (2,2,5,6,6,6,7,7), (2,3,5,5,6,6,7,7), (2,2,6,6,6,6,6,7) or (2,3,5,6,6,6,6,7).

If P is of type $(2,2,2,7,7,7,7,7)$ with PQ a 7-line, then Q lies on exactly one 7-line, no 3-lines and at most three 6-lines. None of the point types satisfy these conditions.

If P is of type $(2,2,3,6,7,7,7,7)$, $(2,3,3,5,7,7,7,7)$ or $(2,3,3,6,6,7,7,7)$ with PQ a 3-line, then Q lies on exactly one 3-line, at most two 7-lines and no 5-lines. None of the remaining eleven point types satisfy these conditions.

If P is either of type $(2,2,4,5,7,7,7,7)$ or $(2,2,4,6,6,7,7,7)$ with PQ a 4-line, then Q lies on exactly one 4-line and at most two 7-lines. None of remaining eight point types satisfy these conditions.

If P is either of type $(2,2,5,6,6,6,7,7)$ or $(2,3,5,6,6,6,6,7)$ with PQ a 6-line, then Q lies on at most two 7-lines and at most five lines with strength 5 or more. None of the remaining six point types satisfy these conditions.

If P is of type $(2,2,5,5,6,7,7,7)$ with PQ a 5-line, then Q lies on exactly one 5-line and no 3-lines. None of the remaining four point types satisfy these conditions.

If P is either of type $(2,3,5,5,6,6,7,7)$ or $(2,3,5,5,5,7,7,7)$ with PQ a 5-line, then Q lies on exactly one 5-line. None of the remaining three point types satisfy these conditions.

Hence, all twenty-six points P are of type $(2,2,6,6,6,6,6,7)$. Counting flags (T,g), g a 7-line, yields the final contradiction, $6b_7 = 26$.

This completes the proof of Theorem 2.9.

CHAPTER 4

A BOUND FOR BLOCKING SETS OF REDEI TYPE, PART III

In Chapters 4 and 5, we extend by one the lower bound on the cardinality of blocking sets of Rédei type of finite projective planes of certain non-square orders. In Chapters 2 and 3, Claim 2.11 afforded us a certain luxury. We knew that there exists a unique i-line of maximal strength. We begin by proving a result similar to Claim 2.11. The purpose of Chapter 4 is to prove the following theorem.

THEOREM 4.1. *Let* Π *be a finite projective plane of non-square order* $n = t^2 - e$, $t \geq 9$ *and* $1 \leq e \leq 2t-2$. *Let* $\mathcal{S}$ *be a blocking set of* Π *of Rédei type with* $\mid \mathcal{S} \mid = n + t + 3$. *Then there exists an unique line* l *of* Π *such that* $\mid l \cap \mathcal{S} \mid = t + 3$.

The proof of Theorem 4.1 will be by way of contradiction. So, assume the existence of at least two secants of strength $t+3$.

Remember that the notation and terminology used in Chapters 2 and 3 will also be used in Chapters 4 and 5.

CLAIM 4.2. *Let* g *be an* i*-line with* $2 \leq i \leq t+2$. *If there exists a triangle of* $(t+3)-$*lines, or if all* $(t+3)-$*lines are incident with a common point on* g, *then* $2 \leq i \leq 6$ *and* $\mid L(g^*) \mid = (t+2)(t+3-i)$.

PROOF: Let l be an $(t+3)$-line and $P = g \cap l$. Through each of the $t+2$ points $Q \in (\mathcal{S} \cap l) \setminus \{P\}$ there are exactly $t+3-i$ secants $h \in L(g^*)$. Every secant $h \in L(g^*)$ intersects l at a point $Q \in (\mathcal{S} \cap l) \setminus \{P\}$. Thus, $\mid L(g^*) \mid = (t+2)(t+3-i)$. Using Lemma 2.8 it is easy to complete the proof. *Q.E.D.*

PROOF OF THEOREM 4.1:

Case 1. Assume there exists a triangle of $(t+3)$-lines with point P one of the vertices. By Lemmas 2.2 and 2.4, P lies on exactly $t+3$ secants and $F(P) \leq t^2+2t+4$. Since $t^2+2t+4 < (t+3)(t+3)$, P lies on an i-line g for some $2 \leq i \leq t+2$. Claim 4.2 implies that $i \leq 6$. Let X be a point on g but not in $\mathcal{S}$. Since $t-3 \geq 6$, there are no j-lines through X with $j > t-3$. Lemma 2.5 implies that point X lies on at least three secants. Thus, $(t+2)(t+3-i) = \mid L(g^*) \mid \geq 2(n+1-i)$. But, as is easily checked, this is false if $t \geq 9$ and $i \leq 6$. Therefore, there does not exist a triangle of $(t+3)$-lines.

Case 2. All $(t+3)$-lines pass through a common point P. Let δ equal the number of $(t+3)$-lines and let l represent one of them.

Suppose the existence of an $(t+\beta)$-line h for some β with $\mid \beta \mid \leq 2$. Since $t \geq 9$, Claim 4.2 implies that the secant h does not pass through P. Let $\mathcal{T}$ represent the subset of points of $\mathcal{S}$ that are not on h or any of the $(t+3)$-lines; $\mid \mathcal{T} \mid = n - \delta(t+1) + 2 - \beta$. Set $Q = h \cap l$. Choose any one of the $n+1-t-\beta$ points X on h yet not in $\mathcal{S}$. By Lemma 2.5, X is either of type $(4-\beta, t+\beta)$ or lies on at least three secants. There are at most

$$\gamma := \lfloor \frac{n - \delta(t+1) + 2 - \beta}{3 - \beta} \rfloor$$

points $X \in h \backslash \mathcal{S}$ of type $(4-\beta, t+\beta)$ such that the $(4-\beta)$-line through X also passes through P. Thus, there are at least $(n+1-t-\beta)-\gamma$ secants of $L(h^*)$ that intersect l at points in $\mathcal{S}\backslash\{P,Q\}$. There are exactly $(t+1)(3-\beta)$ secants in $L(h^*)$ that intersect l at points in $\mathcal{S} \setminus \{P,Q\}$. Thus, it must be that $(t+1)(3-\beta) \geq (n-t-\beta+1)-\gamma$.

Then the following inequalities must be valid:

$$(t+1)(3-\beta)^2 \geq (n-t-\beta+1)(3-\beta)-n+\delta(t+1)-2+\beta$$

$$(*) \qquad (t+1)(3-\beta)^2 \geq n(2-\beta)-(t+\beta-1)(3-\beta)+\delta(t+1)-2+\beta$$

$$(t+1)(3-\beta)^2 \geq (t^2-2t+2)(2-\beta)-(t+\beta-1)(3-\beta)+2t+\beta$$

$$0 \geq (t^2-(7-\beta)t-1)(2-\beta).$$

Thus, if $\beta < 2$, then $7-\beta \geq t^2-1$. This contradiction proves that there do not exist secants h with $t-2 \leq st(h) \leq t+1$.

If $\beta = 2$, then inequality $(*)$ yields $(2-\delta)(t+1) \geq 0$. Hence, $\delta = 2$ and $\mid \mathcal{T} \mid = n-2t-2$. Every point in $(l \cap \mathcal{S}) \setminus \{P, Q\}$ lies on exactly one line of $L(h^*)$. Thus, $\mid L(h^*) \mid$ is the sum of $t+1$ and the number ϵ of lines of $L(h^*)$ through P. Every point X in $h \setminus \mathcal{S}$ is of type $(2, t+2)$, so

$$n-t-1 = h^* = \mid L(h^*) \mid = t+1+\epsilon.$$

Hence, $\epsilon = n-2t-2 = \mid \mathcal{T} \mid$. Thus, if there exists a $(t+2)$-line, then P lies on two $(t+3)$-lines, $n-t-2$ secants of strength 2 and no other secants.

Still under the supposition that there is a $(t+2)$-line h, let g represent an i-line not through P with $2 \leq i \leq t-3$. Let $\mathcal{V}$ be the subset of points of $\mathcal{S}$ that are not on g or the two $(t+3)$-lines; $\mid \mathcal{V} \mid = n-t-i$. If $X \in g \setminus \mathcal{S}$, then X is either on a 2-line or a tangent through P. There are exactly $n-t-i$ points $X \in g \setminus \mathcal{S}$ each lying on a 2-line through P and, since $(t-3)+2-2 < t+2$, Lemma 2.5 implies that each such X lies on at least three secants, one of which does not pass through P and is not g. The $t+1$ points $X \in g \setminus \mathcal{S}$ each lying on a tangent through P each lies on at least one secant not through P and not g. Thus, there are at least $n+1-i$ secants in

$L(g^*)$ not passing through P. There are exactly $(t+1)(t+3-i)$ secants in $L(g^*)$ not passing through P. It is straightforward to verify that $n+1-i > (t+1)(t+3-i)$ for $i \geq 7$. Hence, $2 \leq i \leq 6$. If $3 \leq i \leq 6$, then Lemma 2.5 implies that each of the $t+1$ points $X \in g \setminus \mathcal{S}$ lying on a tangent through P lies on at least two secants not through P which are not g. Again, it is straightforward to convince oneself that $(n-t-i)+2(t+1) > (t+1)(t+3-i)$ for $i \geq 5$. So, $2 \leq i \leq 4$. One last application of Lemma 2.5 shows that each of the $t+1$ points $X \in g \setminus \mathcal{S}$ lying on a tangent through P lies on at least three secants not through P which are not g, and each of the $n-t-i$ points $X \in g \setminus \mathcal{S}$ lying on a 2-line through P lies on at least three secants not through P which are not g. But, $3(n+1-i) > (t+1)(t+3-i)$ for $i \geq 3$. Therefore, if there exists a $(t+2)$-line, then all secants not through P are either 2-lines or $(t+2)$-lines. Let $P_0 \in l \cap \mathcal{S} \setminus \{P\}$. Then P_0 lies on either $t-2$ or $t-3$ lines of strength $t+2$. This implies the existence of a secant f not through P with $3 \leq st(f) \leq t+1$, a contradiction.

Therefore, there are no $(t+2)$-lines.

To complete the proof of the theorem, let g be an i-line through P with $2 \leq i \leq 6$. As there are no secants h such that $t-2 \leq st(h) \leq t+2$, Lemma 2.5 implies that each of the $n-1+i$ points $X \in \mathcal{S} \setminus l$ lies on at least three secants. Thus, we must have that $(t+2)(t+3-i) = \mid L(g^*) \mid \geq 2(n+1-i)$. As is easily verified, this is not true for $t \geq 9$. Thus, P lies only on secants of strength $t+3$. Thus, $\delta(t+2)+1 = n+t+3$. As is easily verified, $\delta = t-2$ or $t-1$. Since every secant intersects with every $(t+3)$-line at a point in $\mathcal{S}$, there exist δ-lines. But, as shown, there do not exist $(t+\beta)$-lines with $\mid \beta \mid \leq 2$.

Q.E.D.

CHAPTER 5

A BOUND FOR BLOCKING SETS OF REDEI TYPE, PART IV

The purpose of Chapter 5 is to prove the following theorem.

THEOREM 5.1. *Let* Π *be a projective plane of non-square order* $n = t^2 - e$, $t \geq 12$ *and* $1 \leq e \leq t$. *Let* $\mathcal{S}$ *be a blocking set of* Π *of Rédei type. Then,* $|\, \mathcal{S} \,| \geq n + t + 4$.

By Theorem 2.9, $|\, \mathcal{S} \,| \geq n + t + 3$. Hence, we will assume $|\, \mathcal{S} \,| = n + t + 3$ and derive a contradiction. By Theorem 4.1, there exists an unique $(t + 3)$-line which will be denoted as l. Note that by Lemma 2.4, if $P \in \mathcal{S} \setminus l$, then $t^2 + t + 5 \leq$ F$(P) \leq t^2 + 2t + 4$.

CLAIM 5.2. *Let* A *be a point in* $\mathcal{S}$ *and on* l. *Then* A *cannot lie on* n *secants of strength 2.*

PROOF: Assume, by way of contradiction, that A does lie on n secants of strength 2. Then the set $\mathcal{S} \setminus \{A\}$ is a blocking set of Rédei type of cardinality $n + t + 2$, a contradiction to Theorem 2.9.

Q.E.D.

CLAIM 5.3. *Each point* $P \in \mathcal{S} \setminus l$ *lies on a secant of strength* $t - 1$ *or more.*

PROOF: Assume not. Then,

$$t^2 + t + 5 \leq F(P) \leq (t + 3)(t - 2) = t^2 + t - 6.$$

Q.E.D.

Lemma 2.6 and Claim 5.3 imply that each point $P \in \mathcal{S} \setminus l$ lies on a secant of strength 5 or less.

CLAIM 5.4. *Each point $P \in \mathcal{S} \setminus l$ lies on a secant of strength t or more.*

PROOF: Assume not and let P be of type α with $\alpha_{t+3} = t - 1$. Since $\alpha_1 \leq 5$,

$$t^2 + t + 5 \leq F(P) \leq (t+2)(t-1) + 5 = t^2 + t + 3.$$

Q.E.D.

Lemma 2.6 now implies that every $P \in \mathcal{S} \setminus l$ lies on a secant of strength 4 or less.

CLAIM 5.5. *Let $P \in \mathcal{S} \setminus l$ be a point of type α. If $\alpha_{t+3} = t$, then $\alpha_1 = 2$.*

PROOF: Assume, by way of contradiction, that $P \in \mathcal{S} \setminus l$ is of type α with $\alpha_{t+3} = t$ and $\alpha_1 \geq 3$. Let $Q \in \mathcal{S} \setminus l$ be such that $PQ = l_1$. By Claim 5.4, there exists a secant g through Q of strength at least t. Since

$$t^2 + 3 \times 4 < t^2 + t + 5,$$

$\alpha_3 \geq 5$. Since $\alpha_3 \geq 5$ and $\alpha_1 \geq 3$, secant g intersects with each of the $t+3$ secants through P at a point in $\mathcal{S}$, a contradiction.

Q.E.D.

CLAIM 5.6. *Let $P \in \mathcal{S} \setminus l$ be a point of type α. Then $\alpha_{t+4-i} \geq t + 5 - i$ for $\alpha_{t+3} - t + 5 \leq i \leq 2t - \alpha_{t+3}$.*

PROOF: Assume, by way of contradiction, that $\alpha_{t+4-i} < t+5-i$ for some i. Define the constant γ by $\alpha_{t+3} = t + \gamma$. By Claim 5.4, γ=0, 1 or 2. Thus,

$$\begin{aligned} F(P) &\leq (i-1)(t+\gamma) + (t+3-i)(t+4-i) + \alpha_1 = \\ &= i^2 - i(t+7-\gamma) + t^2 + 6t + 12 - \gamma + \alpha_1. \end{aligned}$$

If $\gamma + 5 \leq i \leq t - \gamma$, this inequality yields the contradiction $F(P) < t^2 + t + 5$.

Q.E.D.

CLAIM 5.7. *Each point $P \in \mathcal{S} \setminus l$ lies on a secant of strength $t+1$ or $t+2$.*

PROOF: Assume not and let P be of type α with $\alpha_{t+3} = t$. By Claim 5.5, α_1=2. Since

$$(t+1)t + 2 \times 2 = t^2 + t + 4 < t^2 + t + 5,$$

one can choose point $Q \in \mathcal{S} \setminus l$ of type β such that $PQ = l_2$. Claim 5.6 implies that Q does not lie on a secant $g \neq l_2$ with $5 \leq st(g) \leq t$. Since

$$t^2 + 2 \times 3 + 2 = t^2 + 8 < t^2 + t + 5,$$

$\alpha_3 \geq 4$ and, hence, Q does not lie on a secant $g \neq l_2$ of strength $t+1$. That is, Q lies only on secants $g \neq l_2$ with $st(g) = 2$, 3, 4 or $t+2$. Since

$$(t-3)(t+2) + 5 \times 4 + \alpha_2 = t^2 - t + 14 + \alpha_2 < t^2 + t + 5,$$

Q lies on at least $t-2$ secants of strength $t+2$. This implies that $\alpha_3 \geq t-1$.

Suppose $\alpha_3 = t$. If Q lies on fewer than $t-1$ secants of strength $t+2$, then Q lies necessarily on at least $\lceil \frac{t+1}{3} \rceil \geq 5$ secants $h \neq l_2$ of strength at most 4. Hence, Q lies on at least $(t-2)+5+1 = t+4$ secants, a contradiction. Therefore, Q lies on exactly $t-1$ secants of strength $t+2$. Computing one gets,

$$F(P) = (t+1)t + \alpha_2 + 2 = t^2 + t + \alpha_2 + 2$$

and

$$F(Q) = (t-1)(t+2) + \beta_4 + \beta_3 + \beta_2 + \beta_1 = t^2 + t - 2 + \beta_4 + \beta_3 + \beta_2 + \beta_1.$$

Or, $\alpha_2 + 4 = \beta_4 + \beta_3 + \beta_2 + \beta_1$. Since α_2 equals β_j for some $1 \leq j \leq 4$, we have a contradiction. So, $\alpha_3 \neq t$.

Suppose $\alpha_3 = t-1$. It is straightforward to show that $\alpha_2 \geq 4$. Choose $R \in \mathcal{S} \setminus l$ such that PR=l_{t+3} and $st(QR) = t+2$. As $\alpha_3 = t-1$, point R can lie on at most $t-2$ secants of strength $t+2$. Since $\alpha_2 \geq 4$ and $\alpha_3 = t-1$, point R lies only on secants $g \neq l_{t+3}$ with $st(g) = 2, 3, 4, 5$ or $t+2$. Since Q lies on $t-2 > 6$ secants of strength $t+2$, all secants h with $st(h) = 3, 4$ or 5 must pass through Q. Thus, R lies only on secants $g \neq l_{t+3}$ with $st(g) = 2$ or $t+2$. Thus,

$$t^2 + t + 5 \leq F(R) \leq (t-2)(t+2) + t + 4 \times 2 = t^2 + t + 4,$$

a contradiction.

Q.E.D.

Lemma 2.6 now implies that every point $P \in \mathcal{S} \setminus l$ lies on a secant of strength 2 or 3.

CLAIM 5.8. *Let $P \in \mathcal{S} \setminus l$ be of type α. Then $\alpha_4 \geq 3$.*

PROOF: Assume not and let $P \in \mathcal{S} \setminus l$ be of type α with $\alpha_4 = 2$. Let $Q \in \mathcal{S} \setminus l$ be of type β such that $PQ = l_6$. It is easily shown that $\alpha_5 = t+1$ or $t+2$ and that P lies on at least $t-2$ secants of strength $t+2$. Thus, Q lies only on secants $g \neq l_6$ with st(g)=2, $t-1$ or t. If Q lies on fewer than $t+1$ secants $g \neq l_6$ with st$(g) > 2$, then

$$t^2 + t + 5 \leq F(Q) \leq 4t + (t-4)(t-1) + \alpha_6 + 2 \times 2 =$$
$$= t^2 + 10.$$

Thus, Q lies on one $(t+2)$-line, one 2-line and $t+1$ secants of strength t or $t-1$. Choose an $R \in \mathcal{S} \setminus l$ different from Q such that $PR = l_6$. Clearly, R must also lie

on $t+1$ secants of strength t or $t-1$. But, $\beta_2 \geq t-1$ and Lemma 2.2 imply that R can not lie on a secant of strength t or $t-1$, a contradiction.

Q.E.D.

CLAIM 5.9. *Let $P \in \mathcal{S} \setminus l$ be of type α. Then $\alpha_3 \geq 3$.*

PROOF: Assume, by way of contradiction, that there is a point $P \in \mathcal{S} \setminus l$ of type α such that $\alpha_3 = 2$. By Claim 5.8, there is a point $Q \in \mathcal{S} \setminus l$ of type β such that $PQ = l_4$. Claim 5.6 implies that Q does not lie on a secant $g \neq l_4$ with $7 \leq st(g) \leq t-2$. Since

$$(t-2)(t+2) + 2 \times 5 + 3 \times 2 = t^2 + 12 < t^2 + t + 5,$$

$\alpha_5 \geq 6$ and, hence, Q does not lie on a secant $g \neq l_4$ of strength $t-1$. Since

$$5(t+2) + (t-6)(t-1) + \alpha_4 + 3 \times 2 = t^2 - 2t + 22 + \alpha_4 < t^2 + t + 5,$$

$\alpha_{t-2} \geq t$ and, hence, Q does not lie on a secant $g \neq l_4$ of strength 5 or 6. Thus, Q does not lie on any secants $g \neq l_4$ with $5 \leq st(g) \leq t-1$. Since $\alpha_3 = 2$, any $(t+2)$-line must pass through P. Thus,

$$t^2 + t + 5 \leq F(Q) \leq 3(t+1) + (\alpha_5 - 4)t + \alpha_4 + (t + 3 - \alpha_5)4 =$$

$$= \alpha_5(t-4) + 3t + 15 + \alpha_4,$$

so $\alpha_5 \geq t+1$.

Suppose $\alpha_5 = t+2$. Then Q lies only on secants $g \neq l_4$ with $st(g) = 2$, t or $t+1$. Also, note that $\alpha_5 = t+2$ implies the non-existence of 3-lines or 4-lines, except

for possibly l_4. If Q were to lie on fewer than $t+1$ secants $g \neq l_4$ with $st(g) \geq t$, then Q would necessarily lie on at least $t-1$ secants of strength 2 and, hence, as is easily shown, $F(Q) < t^2+t+5$. Thus, Q lies on $t+1$ secants $g \neq l_4$ of strength t or $t+1$, secant l_4 and one 2-line. Let $A \in \mathcal{S} \cap l$ be such that $st(QA) = 2$. As A is not on l_4, there are no 3-lines or 4-lines through A. As $\beta_2 \geq 3$ and $\beta_3 \geq t$, there are no secants g through A with $5 \leq st(g) \leq t+2$. Thus, A lies on n secants of strength 2, a contradiction of Claim 5.2. So, $\alpha_5 \neq t+2$. Thus, $\alpha_5 = t+1$.

Then Q lies only on secants $g \neq l_4$ for which $st(g) = 2$, 3, t or $t+1$, and there are no 4-lines with the possible exception of l_4. The equality $\alpha_5 = t+1$ implies that there are at least two secants $h \neq l_4$ through Q of strength 2 or 3. If Q lies on fewer that t secants $g \neq l_4$ with $st(g) \geq t$, then Q must lie on at least $\lceil \frac{t-1}{2} \rceil \geq 6$ secants $h \neq l_4$ of strength 2 or 3 and, hence, as is easily shown, $F(Q) < t^2+t+5$. Thus, Q lies on t secants $g \neq l_4$ with $st(g) = t$ or $t+1$, secant l_4 and two secants of strengths 2 or 3. Thus,

$$t^2+t+5 \leq F(Q) \leq 3(t+1)+(t-3)t+6+\alpha_4 = t^2+9+\alpha_4,$$

so $\alpha_4 \geq t-4 > 5$.

Let $R \in \mathcal{S} \setminus l$ be of type γ such that $R \notin l_4$. Since $\alpha_5 = t+1$ and $\alpha_4 \geq t-4 > 5$, point R lies only on secants $g \neq PR$ such that $st(g) = 2$, 3, t or $t+1$. If R lies on fewer than t secants $g \neq PR$ with $st(g) \geq t$, then R necessarily lies on at least $\lceil \frac{t-2}{2} \rceil \geq 5$ secants $h \neq PR$ of strength 2 or 3 and, thus, $F(R) < t^2+t+5$. By the contradiction, $\alpha_4 = t+1$. Thus, Q lies on $t+1$ secants of strength t or $t+1$. As Q lies on at most four $(t+1)$-lines, one has that $\beta_3 = t$. Choose $U \in \mathcal{S} \setminus l$ such that $st(QU) = t$. Since $\beta_3 = t$, U lies only on secants $g \neq QU$ for which $st(g) = 2$, 3, 4,

$t+1$ or $t+2$. Since $\alpha_3 = 2$, U lies on at most one $(t+2)$-line and at most four secants of strength $t+1$ or $t+2$. Thus,

$$F(U) \le (t+2) + 3(t+1) + t + (t-3)4 + 3 = 9t - 4 < t^2 + t + 5.$$

So, $\alpha_5 \ne t+1$.

This contradiction completes the proof of Claim 5.9.

Q.E.D.

CLAIM 5.10. *Let $P \in \mathcal{S} \setminus l$ be of type α. Then $\alpha_1 + \alpha_2 \ge 6$.*

PROOF: By way of contradiction, assume the existence of a point P of type α with $\alpha_1 + \alpha_2 \le 5$.

By Claim 5.9, there exists a point $Q \in \mathcal{S} \setminus l$ of type β such that $PQ = l_3$. Claim 5.6 implies that Q does not lie on a secant $g \ne l_3$ with $7 \le st(g) \le t-2$. Since

$$(t-2)(t+2) + 3 \times 5 + \alpha_1 + \alpha_2 \le t^2 + 16 < t^2 + t + 5,$$

$\alpha_5 \ge 6$ and, hence, all secants $g \ne l_3$ through Q of strength $t-1$ or more intersect with l_5 at points of $\mathcal{S}$. Thus, Q does not lie on a secant $g \ne l_3$ of strength $t-1$. As

$$4(t+2) + (t-3)(t-1) + 5 = t^2 + 16 < t^2 + t + 5$$

and

$$5(t+2) + (t-4)(t-2) + 5 = t^2 - t + 23 < t^2 + t + 5,$$

$\alpha_{t-1} \ge t$ and $\alpha_{t-2} \ge t-1$, respectively. Hence, Q also does not lie on secants $g \ne l_3$ of strengths 5 or 6. Thus, Q does not lie on a secant $g \ne l_3$ with $5 \le st(g) \le t-1$.

Note that $\alpha_1+\alpha_2 \leq 5$ implies that Q lies on at most two $(t+2)$-lines. (If $st(l_3) = t+2$, then $F(P) > t^2 + 2t + 4$.) Thus,

$$t^2 + t + 5 \leq F(Q) \leq 2(t+2) + (\alpha_5 - 3)(t+1) + (t + 3 - \alpha_5)4 + \alpha_3 =$$

$$= \alpha_5(t-3) + 3t + 13 + \alpha_3.$$

So, $\alpha_5 \geq t$. Note that if $\alpha_5 = t$, then $\alpha_3 \geq t - 8 \geq 4$, and since Q lies on a secant of strength less than 4, one has

$$t^2 + t + 5 \leq F(Q) \leq 2(t+2) + (t-3)(t+1) + 2 \times 4 + \alpha_3 + 3 =$$

$$= t^2 + 12 + \alpha_3.$$

So, if $\alpha_5 = t$, then $\alpha_3 \geq t - 7 \geq 5$.

Suppose $\alpha_5 = t + 2$. Except for possibly l_2, l_3 and l_4, there are no 3-lines or 4-lines. Thus, Q lies only on secants $g \neq l_3$ with $st(g) = 2,\ t, t+1$ or $t+2$. If Q lies on fewer than $t+1$ secants $g \neq l_3$ with $st(g) \geq t$, then Q necessarily lies on at least $t-1$ secants of strength 2 and, hence, $F(Q) < t^2 + t + 5$. So, Q lies on $t+1$ secants $g \neq l_3$ with $st(g) \geq t$. If $\alpha_4 \geq 5$, then these $t+1$ secants $g \neq l_3$ would each intersect l_4 at a point of $\mathcal{S}$, implying that $\alpha_4 = t+2$. But, $\alpha_4 = t+2$ implies that $F(P) > t^2 + 2t + 4$, a contradiction. So, if $\alpha_5 = t+2$, then $\alpha_4 \leq 4$.

Let $R \in \mathcal{S} \setminus l$ be such that $PR = l_5$. Point R lies only on secants of strengths $2,\ t-1, t, t+1$ or $t+2$. If R does not lie on an $(t-1)$-line, then since $\alpha_4 \leq 4$ one has that

$$F(R) \leq 3(t+2) + (t+1) + t + (t-2)2 = 7t + 3 < t^2 + t + 5.$$

Let g be a $(t-1)$-line. There are $n-t-2$ points $X \in g \setminus \mathcal{S}$ such that $X \notin l_j$, $j = 1, ..., t+3$. Since there are no 5-lines and all 3-lines and 4-lines pass through P, Lemma 2.5 implies that each of the points X is of type $(2,2,2,2,t-1)$. Since $4(n-t-2) > 2n$, there is some point $U \in \mathcal{S} \setminus l$ which lies on at least three 2-lines. This contradiction of Claim 5.8 implies that $\alpha_5 \neq t+2$.

Suppose $\alpha_5 = t$. Remember that $\alpha_3 \geq 5$. Let $R \in \mathcal{S} \setminus l$ be of type γ with $PR = l_4$. Together, $\alpha_5 = t$ and $\alpha_3 \geq 5$ imply that R does not lie on a secant $g \neq l_4$ with $5 \leq st(g) \leq t$. If R were to lie on fewer than $t-1$ secants $g \neq l_4$ with $st(g) \geq t+1$, then

$$F(R) \leq 2(t+2) + (t-4)(t+1) + \alpha_4 + 3 \times 4 + 3 =$$

$$= t^2 - t + 15 + \alpha_4 < t^2 + t + 5.$$

Thus, R lies on exactly $t-1$ secants $g \neq l_4$ with $st(g) \geq t+1$. Thus, $\alpha_3 = t$. Note that R must lie on three secants h that are of strength 4 or less. Let $U \in \mathcal{S} \setminus l$ be such that $PU = l_4$, yet $U \neq R$. Then U as R lies on exactly $t-1$ secants $g \neq l_4$ with $st(g) \geq t+1$. Thus,

$$(\gamma_3 - 1) + (\gamma_2 - 1) + (\gamma_1 - 1) \geq t - 1,$$

a contradiction since $\gamma_3 \leq 4$. Thus, $\alpha_5 \neq t$ and, so $\alpha_5 = t+1$.

Except for possibly l_3 and l_4, there are no 4-lines. Thus, Q lies only on secants $g \neq l_3$ with strengths 2, 3, $t, t+1$ or $t+2$. This fact and the equality $\alpha_5 = t+1$ imply that Q lies on at least two secants $h \neq l_3$ with strengths 2 or 3. If Q lies on fewer than t secants $g \neq l_3$ with $st(g) \geq t$, then Q necessarily lies on at least $\lceil \frac{t-1}{2} \rceil \geq 6$ secants with strength 3 or less and, hence, as is easily shown, $F(Q) < t^2 + t + 5$. So,

Q lies on exactly t secants $g \neq l_3$ with $st(g) \geq t$. If $\alpha_4 \leq 4$, then Q lies on at most three secants of strength $t+1$ or more and, hence,

$$F(Q) \leq 2(t+2) + (t+1) + (t-3)t + 10 = t^2 + 15 < t^2 + t + 5.$$

Thus, $\alpha_4 \geq 5$. By Lemma 2.2, the t secants $g \neq l_3$ through Q of strength t or more intersect with l_4 in points of $\mathcal{S}$. Thus, $\alpha_4 = t+1$ and Q lies only on secants $g \neq l_3$ of strengths 2, 3, $t+1$ or $t+2$.

Assume, by way of contradiction, the existence of a secant $g \neq l_3$ of strength t. Choose a point $R \in \mathcal{S} \setminus l$ of type γ such that $R \notin (l_2 \cup l_3 \cup g)$. As $\alpha_4 = t+1$, R lies only on secants of strengths $2, 3, t, t+1$ or $t+2$. As with Q, point R lies on t secants $h \neq PR$ of strength t or more. Since st(PR)$\geq t+1$, $\gamma_3 \geq t$. This inequality yields the contradiction that g meets $t+1$ of the secants through R in points of $\mathcal{S}$. Hence, there are no t-lines, except for possibly l_3.

Thus, any point $R \in \mathcal{S} \setminus l$ such that $R \notin l_2 \cup l_3$ lies on $t+1$ secants of strengths $t+1$ or $t+2$. This yields the contradiction

$$t^2 + 2t + 4 \geq F(R) \geq (t+1)(t+1) + 2 \times 2 = t^2 + 2t + 5.$$

Thus, $\alpha_5 \neq t+1$.

Therefore, $\alpha_1 + \alpha_2 \geq 6$.

Q.E.D.

CLAIM 5.11. *There are no 3-lines and no $(t+1)$-lines.*

PROOF: Assume, by way of contradiction, that there does exist a secant of strength 3. By Claim 5.10, there is a point $P \in \mathcal{S} \setminus l$ of type α with $\alpha_1 = 3$. Let $Q \in \mathcal{S} \setminus l$ be such that $PQ = l_1$. Point Q lies on a secant g of strength $t+1$ or $t+2$. Since

$\alpha_1 = 3$, all secants of strength $t+2$ pass through P. It is clear that $\alpha_3 = 3$, for else $st(g) \geq t+3$ by Lemma 2.2. If $\alpha_{t+3} = t+2$, then the $n-t-1$ points $X \in l_{t+3} \setminus \mathcal{S}$ are each of type $(2, t+2)$. Thus, one of the $n-t-1$ points in $\mathcal{S} \setminus (l_{t+3} \cup l)$ lies either on two 2-lines or a 2-line and a 3-line, both not possible by Claim 5.10. Thus, $\alpha_{t+3} = t+1$ and there are no secants of strength $t+2$. Claim 5.6 now implies that Q does not lie on secants $g \neq l_1$ such that $6 \leq \mathrm{st}(g) \leq t-1$. Since

$$(t-2)(t+2) + 2 \times 4 + 9 = t^2 + 13 < t^2 + t + 5,$$

$$4(t+2) + (t-1)(t+2) + 9 = t^2 - t + 21 < t^2 + t + 5$$

and

$$3(t+2) + (t-3)t + 9 = t^2 + 15 < t^2 + t + 5,$$

$\alpha_5 \geq 5$, $\alpha_{t-1} \geq t$ and $\alpha_t \geq t+1$. Hence, Q also does not lie on a secant $g \neq l_1$ of strength t, 5 or 4. Thus, Q lies only on secants of strength 3 or $t+1$. Thus, Q must lie on t secants of strength $t+1$ and three secants of strength 3. Thus, $\alpha_4 = t+1$. Let $R \in \mathcal{S} \setminus l$ be of type γ with $PR = l_4$. Point R lies only on secants of strength 2, 3, t or $t+1$. Also, R lies on at least two secants of strengths 3 or less since $\alpha_{t+3} = t+1$. If R lies on fewer than t secants $g \neq l_4$ with $st(g) \geq t$, then R must lie on at least $\lceil \frac{t-1}{2} \rceil \geq 6$ secants of strength 2 or 3, which is easily shown to imply $F(R) < t^2 + t + 5$. Thus, R lies on $t+1$ secants of strength t or more. Thus, either $\gamma_1 + \gamma_2 \leq 5$ or $\gamma_1 = \gamma_2 = 3$ and $\gamma_3 > 3$, both of which have been eliminated as possibilities. Thus, there are no 3-lines.

Since there are no 3-lines, there are no $(t+1)$-lines. For otherwise, if g is a $(t+1)$-line, then the $n-t$ points $X \in g \setminus \mathcal{S}$ are each of type $(2, 2, t+1)$ and, thus, one of the $n-t$ points $P \in \mathcal{S} \setminus (l \cup g)$ lies on at least two 2-lines, a contradiction of Claim 5.10. *Q.E.D.*

Claim 5.12. *Let $P \in \mathcal{S} \setminus l$ be of type α. Then, $\alpha_2 \geq 7$.*

Proof: Assume, by way of contradiction, that there is a point $P \in \mathcal{S} \setminus l$ of type α with $\alpha_2 = 4$, 5 or 6. By Claims 5.7 and 5.11, $\alpha_1 = 2$. Let $A \in \mathcal{S} \cap l$ be such that $PA = l_1$. Let $Q \in \mathcal{S} \setminus l$ be of type β such that $PQ = l_2$. Claim 5.6 implies that Q does not lie on a secant $g \neq l_2$ with $7 \leq st(g) \leq t-2$. Since

$$5(t+2)+(t-4)(t-2)+8 = t^2-t+26 < t^2+t+5,$$

$\alpha_{t-2} \geq t-1$ and, hence, Q does not lie on a 6-line $g \neq l_2$. Thus, Q lies only on secants $g \neq l_2$ of strength $2,4,5,t-1,t$ or $t+2$. (Remember, there are no 3-lines or $(t+1)$-lines.) Computations of F(P) show that $\alpha_6 \geq 7$ and $\alpha_5 \geq 5$. Thus,

$$t^2+t+5 \leq F(Q) \leq (\alpha_6-1)(t+2)+(t+2-\alpha_6)5+\alpha_2+2 =$$

$$= \alpha_6(t-3)+4t+10+\alpha_2.$$

So $\alpha_6 \geq t-1$.

Suppose $\alpha_6 = t-1$. Then Q lies on at most $t-2$ secants of strength $t+2$. If Q lies on fewer than $t-2$ secants of strength $t+2$, then

$$F(Q) \leq (t-3)(t+2)+x+3\times 5+\alpha_2+2 = t^2-t+11+x+\alpha_2,$$

where $x = t$ if $\alpha_2 = 4$, $x = t-1$ if $\alpha_2 = 5$ and $x = 5$ if $\alpha_2 = 6$. In any case, $F(Q) < t^2+t+5$. Thus, Q lies on exactly $t-2$ secants of strength $t+2$. A computation of F(Q) shows that Q lies on at least three 5-lines. Let $g \neq l_2$ be a 5-line through Q. As $\alpha_2 \geq 4$, there exists $R \in (\mathcal{S} \setminus l) \cup l_2$ distinct from P and Q. The $t-2 \geq 10$ secants of strength $t+2$ through R must intersect with g at points

of $\mathcal{S}$, clearly a contradiction since st(g)=5. Since there are no $(t+1)$-lines, $\alpha_6 = t$ or $t+2$.

Suppose $\alpha_6 = t+2$. Then A does not lie on a secant g with $3 \leq st(g) \leq t-2$. Since $\alpha_2 > 2$, A does not lie on an $(t+2)$-line. There are no $(t+1)$-lines. Assume the existence of a secant g through A of strength $t-1$ or t. Since $\alpha_6 = t+2$, all 4-lines and 5-lines, if any, pass through P. There are no 3-lines. Thus, each of the points $X \in g \setminus \mathcal{S}$ is either of type $(2,2,2,2,t-1)$ or $(2,2,2,t)$, respectively. Thus, P lies on at least four 2-lines, a contradiction of Claim 5.8. Thus, point A lies on n secants of strength 2, a contradiction of Claim 5.2. Thus, $\alpha_6 \neq t+2$, so $\alpha_6 = t$.

Point A lies only on secants $g \neq l_1$ of strength 2, 4, $t-1$ or t. Assume the existence of a t-line g through A. Since $\alpha_5 \geq 5$, $\alpha_2 = \alpha_3 = \alpha_4 = 4$. Since

$$4(t+2) + (t-5)t + 3 \times 4 + 2 = t^2 - t + 22 < t^2 + t + 5,$$

there are at least five $(t+2)$-lines through P. Thus, all 4-lines pass through P and each point $X \in g \setminus \mathcal{S}$ is of type $(2,2,2,t)$. Hence, P lies on four 2-lines, a contradiction of Claim 5.8.

Assume the existence of a $(t-1)$-line g through A. Then $\alpha_5 = 5$ and , as is easily shown, there are at least five $(t+2)$-lines through P. Thus, all 4-lines pass through P. Since $\alpha_6 = t$, all 5-lines also pass through P. Hence, each of the points $X \in g \setminus \mathcal{S}$ is of type $(2,2,2,2,t-1)$. Hence, P lies on five 2-lines, a contradiction of Claim 5.8.

Assume the existence of a 4-line g through A. Then there are at most three $(t+2)$-lines through P. A computation of F(P) shows that $\alpha_4 > 6$. Let $R \in \mathcal{S} \setminus l$ be such that $PR = l_3$. Since $\alpha_4 > 6$,

$$t^2 + t + 5 \leq F(R) \leq (\alpha_2 - 1)(t+2) + (t + 2 - \alpha_2)4 + \alpha_3 + 2 =$$

$$= \alpha_2(t-2) + 3t + 8 + \alpha_3,$$

a contradiction since $\alpha_2 \leq 6$. Thus, A lies on n secants of strength 2, a contradiction of Claim 5.2.

Therefore, $\alpha_2 \geq 7$.

Q.E.D.

Therefore, there are no secants of strength 3, 4, 5 or 6.

PROOF OF THEOREM 5.1:

Let $P \in \mathcal{S} \setminus l$ be of type α. Then $\alpha_1 = 2$ and $\alpha_2 \geq 7$. Let $A \in \mathcal{S} \cap l$ be such that $PA = l_1$. Claim 5.6 and $\alpha_2 \geq 7$ imply that A lies on n secants of strength 2, a contradiction of Claim 5.2.

Q.E.D.

CHAPTER 6

SMALL BLOCKING SETS OF HERMITIAN DESIGNS, PART I

Motivation for the next two chapters began with a paper by D. Jungnickel [15]. In his work Jungnickel introduces the idea of a *self blocking* block design. Given any block design Σ that admits blocking sets, form the collection $\mathcal{C}(\Sigma)$ of all committees of Σ. (A blocking set of minimum cardinality is called a *committee*.) If $\mathcal{C}(\Sigma)$ is itself a block design, where the points of $\mathcal{C}(\Sigma)$ are the points of Σ and the blocks of $\mathcal{C}(\Sigma)$ are the committees of Σ, then Σ is said to be *self blocking* when the committees of the block design $\mathcal{C}(\Sigma)$ are the blocks of Σ. That is, Σ is self-blocking if $\mathcal{C}(\mathcal{C}(\Sigma)) = \Sigma$.

Let q be a prime power, $K = GF(q)$ the finite field of q elements and V a 3-dimensional vector space over K. One can form a finite projective plane Π in the following manner: Let the 1-dimensional subspaces of V be the points of Π and the 2-dimensional subspaces of V be the lines of Π. As is well known, Π is of order q, is denoted by $PG(2,q)$ and is called the Desarguesian projective plane of order q. The points of $\Pi = PG(2,q)$ can be represented by homogeneous triples (x,y,z); that is, (x,y,z) can be thought of as a 1×3 matrix where $(x,y,z) = (kx,ky,kz)$ for all non-zero $k \in K$ and x, y and z are not simultaneously zero. The lines of Π can be represented by the transpose $(a,b,c)^t$ of a homogeneous triple. Then the point $P = (x,y,z)$ lies on line $l = (a,b,c)^t$ if and only if $ax + by + cz = 0$; that is, the "dot product" is zero. [12, Chapter II]

In Jungnickel's paper [15] it is shown that for q a prime power, the projective planes $\Pi = PG(2, q^2)$ are self-blocking; the committees of Π are its Baer subplanes. Inside of Π are substructures called Hermitian unitals which are block designs and, though not committees, are blocking sets. The collection of all Hermitian unitals forms a block design, call it Σ. The original motivating question for Chapters 6 and 7 was for which values of q, if any, are the committees of Σ the lines of the plane. In order to address that question we will need the following definitions.

A *collineation* α of the plane $\Pi = PG(2, q)$, q a prime power, is a map of the set of points of Π onto itself such that point P lies on line l if and only if point $\alpha(P)$ lies on line $\alpha(l)$; that is , α preserves incidence. (Here, $\alpha(l)$ denotes the set of points $\alpha(Q)$, Q a point on l. It is easily verified that $\alpha(l)$ is a line of Π.) Denote by $Aut\Pi$ the group of all collineations of Π. The *projective subgroup* $PGL(3, q)$ of $Aut\Pi$ is the set of all collineations that can be represented by non-singular 3×3 matrices A with entries from the Galois field $GF(q)$. That is, if $P = (x, y, z)$ is a point of Π, then the collineation α represented by A is defined by $\alpha(P) = (x, y, z)A$. (It is easy to convince oneself that for non-zero $k \in K$, kA and A represent the same collineation.) [12, Chapter II]

For positive integer n, a *unital* is a 2-$(n^3 + 1, n + 1, 1)$ block design. For odd prime power q, set $\Pi = PG(2, q^2)$. The plane Π possesses sub-structures which are 2-$(q^3 + 1, q + 1, 1)$ block designs; that is, there are unitals inside Π. Let U represent a unital of Π. Each line of Π is either tangent to U, intersecting with U in exactly one point, or is a secant intersecting with U in exactly $q+1$ points [13, Chapter 6.3]. Thus, every unital of Π is a blocking set of Π. Denote by $U(q)$ the (non-empty) set of all unitals of $\Pi = PG(2, q^2)$, q an odd prime power. The plane Π admits the doubly transitive automorphism group $G = PGL(3, q^2)$, and G preserves unitals, that is, elements of G map unitals to unitals. Thus, $U(q)$ is itself a block design

where its points are the points of Π and its blocks are the unitals of Π. Call $U(q)$ the *unitals design* of Π. [12, Theorem 2.49]

A *correlation* α of a (finite) projective plane Π is a map of the set of points of Π onto the set of lines of Π, and vice versa, such that point P lies on line l if and only if point $\alpha(l)$ lies on line $\alpha(P)$. A *polarity* α is a correlation of order 2; that is, α^2 is the identity collineation. [12, Chapter II.6]

Let $\Pi = PG(2, q^2)$, q an odd prime power. A *unitary polarity* α of Π is a polarity that can be represented by a 3×3 non-singular Hermitian matrix H with entries from $F = GF(q^2)$ in the following manner. (The matrix $H = (h_{ij})$, $h_{ij} \in F$, is Hermitian if and only if $h_{ij} = h_{ji}^q$. That is, $H = (H^q)^t$.) The point $P = (x, y, z)$ is sent by α to the line $H(x^q, y^q, z^q)^t$ and the line $l = (a, b, c)^t$ is sent to the point $(a^q, b^q, c^q)H^{-1}$. Point $P = (x, y, z)$ of Π is said to be an *absolute point* of α if and only if P lies on line $\alpha(P)$; that is, if and only if $(x, y, z)H(x^q, y^q, z^q)^t = 0$. Similarly, line $l = (a, b, c)^t$ is said to be an *absolute line* of α if and only if $(a^q, b^q, c^q)H^{-1}(a, b, c)^t = 0$. The absolute points and non-absolute lines of a unitary polarity form a unital U, a 2-$(q^3 + 1, q + 1, 1)$ block design. (The absolute lines of a unitary polarity α are called *tangents*, because they intersect with U in exactly one point.) Call such a unital a *Hermitian unital* and denote by $H(q)$ the set of all Hermitian unitals of Π. Since the projective group $G = PGL(3, q^3)$ preserves (Hermitian) unitals and its unitary subgroup, the subgroup of G that fixes some Hermitian unital U, has order $(q^3 + 1)q^3(q^2 - 1)$, it is routine to verify that $H(q)$ is a block design with parameters $v = q^4 + q^2 + 1$, $k = q^3 + 1$ and $\lambda = q^4(q^2 - 1)$. Call $H(q)$ the *Hermitian design* of Π. [12, Chapter II.8]

For the remainder of this chapter, let q denote an odd prime power. Then Π will denote $PG(2, q^2)$ and C will denote a committee of $H(q)$. Also, let F be the

Galois field GF(q^2) and K the Galois field GF(q). For real number r, denote the smallest integer greater than or equal to r by $\lceil r \rceil$.

In Chapter 7 it is shown that for $q = 3$ the committees of the Hermitian design $H(q)$ are the lines of Π; that is, $\mathcal{C}(H(3))=\Pi$. Initially it was hoped to extend this result and show that $\mathcal{C}(H(q))=\Pi$ for all values of q. No progress was made in this direction. After contacted by my advisor D. Drake, A. Blokhuis demonstrated that this extension was unattainable. [4] (An outline of Blokhuis' argument will be given later in this chapter.) So, are there any values of q other than 3 for which $\mathcal{C}(H(q))=\Pi$? For which values of q is it true that $\mathcal{C}(H(q))$ is not equal to Π? In attempting to answer these questions, a lower and upper bound are found on the cardinality of a committee $\mathcal{C}$ of $H(q)$.

The following definitions and Lemmas 6.1 through 6.5 will be needed to establish a lower bound on the cardinality of $\mathcal{C}$ and to show in Chapter 7 that $\mathcal{C}(H(3))$ =PG(2,9).

Let V be a vector space over K=GF(q) of dimension 2d. A *spread* of V is a set of $q^d + 1$ d-dimensional subspaces $V_1, ..., V_{q^d+1}$ of V such that $V_i \cap V_j = \{0\}$ for $i \neq j$. [12, Exercise 7.7] The field F=GF(q^2) is a 2-dimensional vector space over K. Since there are exactly $q + 1$ 1-dimensional subspaces of F over K, there is a unique spread Σ_1 of F over K.

Let $L[a, b]$ denote the set $\{x \in F \mid (ax)^q + ax + b = 0, b \in K \text{ and } 0 \neq a \in F\}$.

LEMMA 6.1. *The collection* $\Sigma_1 := \{\ L[a, 0] \mid a \in F, a \neq 0\ \}$ *is a spread of* F *over* K.

PROOF: Since $(x + y)^q = x^q + y^q$ for all $x, y \in F$, and since $L[a, 0] \neq F$, $L[a, 0]$ is a 1-dimensional subspace of F over K. The polynomial $X^q + X$ has $q - 1$ non-zero roots in F. If r is one such root, then given any non-zero $c \in F$, one has

$c \in L[rc^{-1}, 0]$. Clearly, $0 \in L[a, 0]$ for all non-zero $a \in F$. Thus, all vectors $c \in F$ lie in some $L[a, 0]$. As 1-dimensional subspaces are either equal or intersect only at 0, Σ_1 is a spread of F.

Q.E.D.

The Desarguesian affine plane of order q can be represented as the collection of cosets of the (unique) spread Σ_1 of F over K. Henceforth, Σ will denote the affine plane of order q formed from the cosets of Σ_1.

LEMMA 6.2. *Every set $L[a, b]$ is a coset of the subspace $L[a, 0]$ and, hence, a line of the affine plane Σ, and; every coset of $L[a, 0]$ is of the form $L[a, b]$. Further, lines $L[a, b]$ and $L[c, d]$ are parallel if and only if $ac^{-1} \in K$; in particular, $L[a, 0] = L[c, 0]$ if and only if $ac^{-1} \in K$.*

PROOF: For any non-zero $a \in F$, the mapping f_a from F into K defined by $f_a(x) = (ax)^q + ax$ is linear and onto. Thus, there exists $c \in F$ such that $(ac)^q + ac = -b$. (Remember, $b \in K$.) For $x \in L[a, 0]$, one has $(a(x + c))^q + a(x + c) + b = (ax)^q + ax + (ac)^q + ac + b = 0 + 0 = 0$. Thus, $c + L[a, 0] \subseteq L[a, b]$. If $x \in L[a, b]$, then $(a(x - c))^q + a(x - c) = (ax)^q + ax - (ac)^q - ac = -b - (-b) = 0$. Thus, $(x - c) \in L[a, 0]$, or $L[a, b] \subseteq c + L[a, 0]$. Hence, $L[a, b] = c + L[a, 0]$, or $L[a, b]$ is a coset of $L[a, 0]$. If $d + L[a, 0]$ is a coset of $L[a, 0]$, then it is straightforward to check that $d + L[a, 0] = L[a, b]$, where $-b = (ad)^q + ad$.

For all non-zero a, the line $L[a, 0]$ contains the point 0. The line $L[a, b]$ is parallel to the line $L[a, 0]$ for all b. Thus, $L[a, b]$ is parallel to $L[c, d]$ if and only if $L[a, 0] = L[c, 0]$. Assume $ac^{-1} \in K$. Then, $(ac^{-1})^q = ac^{-1}$. Thus, $(cx)^q + cx = 0$ if and only if $ac^{-1}((cx)^q + cx) = 0$ if and only if $(ax)^q + ax = 0$. So, $L[a, 0] = L[c, 0]$.

Assume $L[a,0] = L[c,0]$. If $0 \neq x \in L[c,0]$, then $L[c,0] = \{ \alpha x \mid \alpha \in K\}$. It is straightforward to verify that $ac^{-1}x \in L[c,0]$, so $ac^{-1} \in K$.

Q.E.D.

For c and e elements of F, denote by $H[c,e]$ the Hermitian unital represented by the following matrix.

$$A = \begin{pmatrix} 0 & 0 & c \\ 0 & 1 & e \\ c^q & e^q & e^{q+1} \end{pmatrix}$$

Since A is non-singular, c is non-zero.

LEMMA 6.3. *The point $(r,s,1)$ of the plane Π is in $H[c,e]$ if and only if c is on the line $L := L[r, f(e,s)]$, where $f(e,s) := (e+s^q)^{q+1}$; line L contains the point 0 if and only if $e = -s^q$.*

PROOF: The point $(r,s,1)$ is in $H[c,e]$ if and only if $0 = (cr)^q + cr + e^{q+1} + (es)^q + es + s^{q+1} = (cr)^q + cr + (e+s^q)^{q+1}$ if and only if $c \in L[r,(e+s^q)^{q+1}]$.

Clearly, L contains the point 0 if and only if $f(e,s) = 0$, if and only if $e = -s^q$.

Q.E.D.

Henceforth, let $f(e,s) := (e+s^q)^{q+1}$

LEMMA 6.4. *The lines $(0,0,1)^t$ and $(1,0,0)^t$ of Π are tangent to $H[c,e]$ at the points (1,0,0) and $(0,-e^q,1)$, respectively.*

PROOF: If the matrix A represents $H[c,e]$, then the proof is simply a matter of matrix multiplication once A^{-1} has been determined. Note that $x^{q+1} + e^q x^q + ex + e^{q+1}$ equals $(x+e)^{q+1}$.

Q.E.D.

LEMMA 6.5. *[14, Theorem 1', p.257] Let V be a vector space of dimension n over a finite field K with q elements. Then any covering of the non-zero elements of V with hyperplanes not containing zero must consist of at least $n(q-1)$ hyperplanes.*

We are now in a position to establish a lower bound on the cardinality of a committee $\mathcal{C}$ of $H(q)$.

PROPOSITION 6.6. $|\mathcal{C}| \geq 2q+2$ *for* $q \geq 3$.

PROOF: Assume, by way of contradiction, that $|\mathcal{C}| \leq 2q+1$. Choose distinct lines g and h from Π that maximize $|\mathcal{C} \cap (g \cup h)|$. Thus, $|\mathcal{C} \cap (g \cup h)| \geq 4$. Since $2q+1 < q^2$, there are distinct points $P \in g$ and $T \in h$ not in $\mathcal{C}$ so that the line $l = PT$ has empty intersection with $\mathcal{C}$. Coordinatize Π so that $P = (1,0,0)$, $Q = g \cap h = (0,1,0)$ and $T = (0,0,1)$. Denote the i points in $\mathcal{C} \setminus (g \cup h)$ by $R_j = (r_j, s_j, 1)$, $1 \leq j \leq i \leq 2q-3$. Note that for all j, the elements r_j and s_j are non-zero. Lemma 6.4 implies that for every c in F, the lines g and h are tangent to the unital $H[c,0]$ at the points P and T, respectively. Since $s_j \neq 0$ for all j, Lemma 6.3 implies that none of the j lines $L[r_j, f(0,s_j)]$ contains the element 0 of F. Thus, by Lemma 6.5, the $i \leq 2q-3$ lines $L[r_j, f(0,s_j)]$ do not cover the non-zero elements of F. Hence, there exists a non-zero element c in F such that the unital $H[c,0]$ has empty intersection with $\mathcal{C}$. This contradicts the fact that $\mathcal{C}$ is a blocking set of $H(g)$.

Q.E.D.

There will now be presented three arguments for upper bounds on the cardinality of $\mathcal{C}$. The bounds will be successively lower. The last argument is independent of the first two. The first argument is an outline of the Blokhuis' argument mentioned earlier.

LEMMA 6.7. *Consider four points in standard position and the six lines joining them. (That is, there are four distinct points, no three collinear.) Then these six lines cannot all be tangents of the same Hermitian unital U.*

PROOF: Label the four points as $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$. This completely determines the representations of the six lines. If U is represented by the Hermitian matrix A and $l = (a,b,c)^t$ is any one of these six lines, then l is tangent to U if and only if $(a^q, b^q, c^q)A^{-1}(a,b,c)^t = 0$. Simple matrix computations will show that the six lines can not all be tangents.

Q.E.D.

As is well known, a Baer subplane Γ of Π intersects with each line l of Π either in $q+1$ points or in exactly one point.[12, Theorem 3.7] If Γ intersects with l in $q+1$ points, then $\Gamma \cap l$ is called a *Baer subline* of l.[16, Definition 1]

LEMMA 6.8. *[15, Lemma 2] The lines of Π that intersect with a Hermitian unital in $q+1$ points do so in a Baer subline.*

Let I be an incidence structure and P a point of I. The *internal structure* I_P of I is defined as the set of points of I minus P and the blocks of I that contain P, where the incidence relation of I_P is that of I.

A 3-$(q^2+1, q+1, 1)$ design is called an *inversive plane* of order q. Denote this design by $\Delta(q)$. The blocks of such designs are called *circles*. An inversive plane can be characterized as those (finite) incidence structures I such that the internal structure I_P is an affine plane for every point P in I.[10, Chapter 6.1]

LEMMA 6.9. *[7, Lemma 3.1] The Baer sublines of a line l of Π form an inversive plane.*

Blokhuis then made the following count.

Let $\Delta(q)$ be the inversive plane formed from the Baer sublines of line l of Π. Let n be any positive integer and T any collection of n circles of $\Delta(q)$. In two ways count the ordered pair (T, d), where d represents a circle of $\Delta(q)$ that has disjoint intersection with every circle of T. It is straightforward to show, see [10, Chapter 6.1], that $\Delta(q)$ contains exactly $q^3 + q$ circles and each circle is disjoint from exactly $\frac{q(q-1)(q-2)}{2}$ circles. If α represents the average number of circles having disjoint intersection with every circle in T, then one gets the following.

$$\alpha = (q^3 + q)\left(\frac{q^3 - 3q^2 + 2q}{2}\right)^n \left(\frac{1}{(q^3 + q)^n}\right)$$

Clearly, $\alpha \leq (q^3 + q)2^{-n}$. That is, if $n > \log_2(q^3 + q)$, then $\alpha < 1$. Hence, there exists a hitting set of $\Delta(q)$ with at most $\beta := (q + 1)(\lceil \log_2(q^3 + q) \rceil)$ points. Lemma 6.7 now implies that there exists a blocking set of $H(q)$ that contains at most 6β points. Thus, 6β is an upper bound on the cardinality of $\mathcal{C}$. Since 6β is approximately $18q \log_2 q$, and since it is clear that $18q \log_2(q) < q^2 + 1$ for large enough q, for at least these large enough values of q, $\mathcal{C}(H(q)) \neq \Pi$.

This completes the outline of Blokhuis' argument. The second argument consists of two minor improvements to Blokhuis' argument.

LEMMA 6.10. *Given that δ is a primitive root of $F = GF(q^2)$, the four concurrent lines $(1,0,0)^t$, $(0,1,0)^t$, $(1,1,0)^t$ and $(\delta,1,0)^t$ can not all be tangent to the same Hermitian unital U.*

PROOF: Let U be represented by the Hermitian matrix A. The inverse A^{-1} of A is also Hermitian. Set $A^{-1} = (a_{ij})$. Assume by way of contradiction that the four lines are tangent to U. Simple computations show that $a_{11} = 0$, $a_{22} = 0$, $a_{12} = -a_{21}$

and $a_{21}(\delta - \delta^q) = 0$. If $a_{21} = 0$, then A^{-1} is singular. If $\delta - \delta^q = 0$, then δ is not primitive. In either case there is a contradiction.

Q.E.D.

For any line l of Π, let $\Delta(q)$ be the inversive plane formed from the Baer sublines of l. Call any set of $q+1$ distinct points of $\Delta(q)$ a *pseudo – circle*. Let n be positive integer and T be a collection of n pseudo–circles. In two ways count (T, d), where d represents a circle having disjoint intersection with each of the pseudo–circles of T. If α represents the average number of circles having disjoint intersection with every pseudo-circle of T, then the following is true.

$$\alpha = (q^3 + q) \times \frac{(\binom{q^2-q}{q+1})^n}{(\binom{q^2+1}{q+1})^n}$$

Define R as follows.

$$R = \frac{\binom{q^2-q}{q+1}}{\binom{q^2+1}{q+1}}$$

Then there exists a hitting set of $\Delta(q)$ containing at most $\epsilon := (q+1)n + \lfloor\alpha\rfloor$ points. (Here, $\lfloor\alpha\rfloor$ denotes the greatest integer less than or equal to α.) Lemma 6.10 now implies that there exists a blocking set of $H(q)$ containing at most 4ϵ points. Thus, 4ϵ is an upper bound on the cardinality of $\mathcal{C}$. It is easily seen that $4\epsilon \leq 6\beta$, so that this is a smaller upper bound.

If one chooses integer $n = \lceil -\log_R(q^3 + q)\rceil$, then one has $\alpha < 1$. It is now straightforward to verify that $4((q+1)(\lceil -\log_R(q^3+q)\rceil) < q^2+1$ for $q \geq 47$. Thus, $\mathcal{C}(H(q))$ can not equal Π for $q \geq 47$. In fact, by carefully choosing n, it can be shown that $\mathcal{C}(H(q))$ can not equal Π for $q \geq 25$.

This completes the second argument.

The third and sharpest argument begins with another personal communication from A. Blokhuis. In his communication Blokhuis credits T. Szonyi for pointing out the survey paper by Z. Furedi from which the following lemma by L. Lovasz was taken. [11]

LEMMA 6.11. *[11, Corollary 6.29] Let G be a hypergraph and denote its maximum degree by r. If τ represents the covering number of G and τ^* represents the fractional covering number of G, then $\tau \leq (1 + \frac{1}{2} + ... + \frac{1}{r})\tau^* < (1 + log(r))\tau^*$.*

A *hypergraph* G is just an incidence structure where the points are called *vertices* and the blocks are called *edges*. The hypergraph to which this lemma will be applied is the design $H(q)$. The number r equals the maximum degree of a vertex (point) in G; that is, if $r(p)$ denotes the number of edges (blocks) passing through vertex (point) p, then $r :=$ max $r(p)$, the maximum taken over all vertices p of G. All points of $H(q)$ have the same degree. The *covering number* τ is the minimum cardinality of a hitting set of G. To define τ^*, the *fractional covering number* of G, start by letting t represent any real valued function defined on the set of vertices (points) of G such that $t(p) \geq 0$ for all points p in G and the sum $\Sigma t(p)$, taken over all points in edge (block) E, is greater than or equal to 1 for all edges (blocks) E in G. If $|t| := \Sigma t(p)$, the sum taken over all points p in G, then the fractional covering number $\tau^* := min|t|$, taken over all t.

PROPOSITION 6.12. $|\mathcal{C}| \leq \lceil((q^4 + q^2 + 1)(q^3 + 1)^{-1})(1 + log(q^7 - q^3))\rceil$ *for* $q \geq 5$.

PROOF: Set $\gamma = (q^4 + q^2 + 1)(q^3 + 1)^{-1})$. Recall that the block design $H(q)$ has parameters $v = q^4 + q^2 + 1$, $k = q^3 + 1$ and $\lambda = q^4(q^2 - 1)$. It is straightforward to verify that each point has degree $r = q^7 - q^3$. Define the constant function t on the v points of $H(q)$ by $t(p) = \frac{1}{k}$, where p is a point of $H(q)$. Then $\tau^* \leq |t| = \frac{v}{k} = \gamma$.

By Lemma 6.11, there is a hitting set S of $H(q)$ with at most $\delta := \lceil \gamma(1 + \log r) \rceil$ points. Since $\delta < q^3 + 1$ for $q \geq 5$, S is actually a blocking set of $H(q)$ for $q \geq 5$.

Q.E.D.

It is easily verified that $\delta < q^2 + 1$ for $q \geq 25$. It is just as easy to verify that for $q = 23$, $\lceil (1 + \frac{1}{2} + \ldots + \frac{1}{r})\gamma \rceil < 23^2 + 1$. Hence, by Lemma 6.11, there are blocking sets of the block design $H(q)$ with fewer than $q^2 + 1$ points for $q \geq 23$. Thus, for $q \geq 23$, $\mathcal{C}(H(q)) \neq \Pi$.

CHAPTER 7

SMALL BLOCKING SETS OF HERMITIAN DESIGNS, PART II

In Chapter 6 it was shown that $\mathcal{C}(H(q)) \neq PG(2, q^2)$ for q an odd prime power greater than or equal to 23. In this chapter we prove the following.

THEOREM 7.1. $\mathcal{C}(H(3)) = PG(2,9)$.

The notation and terminology employed in Chapter 6 will also be used here.

Define $F{:=}GF(9)$ as the polynomial ring $GF(3)[X]$ modulo the ideal generated by the irreducible polynomial $X^2 + 1$. The polynomial $X + 1$ is a primitive root of F. Set $\sigma = X + 1$.

Let $\mathcal{C}$ be a committee of the block design $H(3)$. Since every line l of $\Pi = PG(2,9)$ intersects with every unital U of Π, and as l does not contain a unital U, it is clear that $\mathcal{C}$ contains at most 10 points. If $\mathcal{C}$ contained fewer than 10 points, then clearly there would exist a blocking set $\mathcal{B}$ of $H(3)$ with exactly 10 points such that the 10 points are not linear. So, we will assume that $\mathcal{B}$ is a blocking set of $H(3)$ with exactly 10 points, the 10 points not collinear. It will be shown that such a blocking set $\mathcal{B}$ can not exist. Hence, it will follow that the committees of $H(3)$ are the lines of Π.

LEMMA 7.2. *For any line g of Π, $|g \cap \mathcal{B}| \leq 8$.*

PROOF: Assume, by way of contradiction, that $|g \cap \mathcal{B}| = 9$. Let P be the point on g not in $\mathcal{B}$, Q any point on g in $\mathcal{B}$ and T the point in $\mathcal{B}$ not on g. Coordinatize

Π such that $P = (1,0,0)$, $Q = (0,1,0)$ and $T = (0,0,1)$. Then for any $0 \neq e \in$F, Lemma 6.5 implies that the unital $U[c,e]$ is disjoint from $\mathcal{B}$, a contradiction.

Q.E.D.

LEMMA 7.3. *Given distinct lines g and h of Π,* $|(g \cup h) \cap \mathcal{B}| \leq 5$.

PROOF: Assume, by way of contradiction, that there are two distinct lines g and h such that $n := |(g \cup h) \cap \mathcal{B}| \geq 6$, where n is maximum in the sense that $n \geq |(g_1 \cup h_1) \cap \mathcal{B}|$ for any pair of distinct lines g_1 and h_1. Without loss of generality assume $|g \cap \mathcal{B}| \geq |h \cap \mathcal{B}|$ and set $Q = g \cap h$.

If $n = 10$, then by Lemma 7.2 one can choose points P on g and T on h not equal to Q and not in $\mathcal{B}$. Coordinatize Π so that $P = (1,0,0)$, $Q = (0,1,0)$ and $T = (0,0,1)$. By Lemma 6.4, any unital $U[c,0]$ has disjoint intersection with $\mathcal{B}$, a contradiction. Hence, $n \leq 9$.

For $1 \leq i \leq 10 - n$, denote by R_i the points in $\mathcal{B} \setminus (g \cup h)$.

Assume that $n \geq 7$. Let P be any point on g not in $\mathcal{B}$. There are 9 lines other than g passing through P. Since $9 > |h \cap \mathcal{B}| + (10 - n)$, one can choose a point T on h so that the line $l := PT$ is disjoint from $\mathcal{B}$. Coordinatize Π so that $P = (1,0,0)$, $Q = (0,1,0)$, $T = (0,0,1)$ and $R_i = (r_i, s_i, 1)$, $1 \leq i \leq 10 - n$. Since none of the points R_i lie on the line $l = (0,1,0)^t$, not only is r_i non-zero but so is s_i for all i. Thus, none of the lines $L_i := L[r_i, f(0,s_i)]$ of the affine plane Σ contain the point 0. Since $n \geq 7$ implies that $10 - n \leq 3 < 4$, Lemma 6.5 implies that there exists a non-zero $c \in F$ not covered by the lines L_i. Thus, none of the points R_i lie in the Hermitian unital $H[c,0]$. Thus, by Lemma 6.4, $H[c,0]$ has disjoint intersection with $\mathcal{B}$, a contradiction. Hence, $n = 6$.

Since $n = 6$ implies that $|g \cap \mathcal{B}| \leq 4$, and since we are assuming $|h \cap \mathcal{B}| \leq |g \cap \mathcal{B}|$, one can choose points P on g and T on h not in $\mathcal{B}$ and not equal to Q such that

P, T, and R_4 are collinear; denote this line by l. Since n is maximum there is at least one R_i not on l; without loss of generality assume $i = 1$. Coordinatize Π so that $P = (1,0,0)$, $Q = (0,1,0)$, $T = (0,0,1)$ and $R_i = (r_i, s_i, 1)$, $i = 1, ..., 4$. Note that $s_1 \neq 0$ and $s_4 = 0$. For any given e in F, represent the line $L[r_i, f(e, s_i)]$ by L_i, $i = 1, ..., 4$, and define $J_e = L_1 \cup ... \cup L_4$.

By Lemma 6.3, for each of the 5 values of e not equal to $2s_1^3$, $2s_2^3$, $2s_3^3$ or 0, the point 0 of Σ is not contained in any of the lines L_i. Thus, for these 5 values of e, no set of three of these lines L_i constitute a parallel class of the affine plane Σ.

Assume that three or more of the lines L_i are in the same parallel class Δ. Let e be any of the 5 values not equal to $2s_1^3$, $2s_2^3$, $2s_3^3$ or 0. Then the parallel lines constitute at most two lines of Δ since no line L_i contains the point 0. Thus, for each of these 5 values of e, the set J_e contains at most 7 points of Σ. Thus, for each of these 5 values of e, there is a non-zero value $c(e)$ in F so that the unital $U[c(e), e]$ does not contain any of the four points R_i. As $|h \cap B| < 5$, Lemma 6.4 implies that there is a unital $U[c, e]$ which has disjoint intersection with $\mathcal{B}$, a contradiction. Hence, no three of the lines L_i are parallel.

Assume that the four lines L_i are pairwise non-parallel. Again, let e be any of the 5 values not equal to $2s_1^3$, $2s_2^3$, $2s_3^3$ or 0. If for any of these 5 values of e three of the four lines L_i form a triangle, then the set J_e contains at most 7 points of Σ. If for any of these 5 values of e the 4 lines are concurrent, then since none of the lines L_i contains the point 0 two of the lines must be equal and, hence, J_e again contains at most 7 points of Σ. In either case, for each of these 5 values of e there is a non-zero $c(e)$ in F so that $U[c(e), e]$ contains none of the points R_i. Since $|h \cap B| < 5$, Lemma 6.3 implies the existence of a unital which has disjoint intersection with $\mathcal{B}$, a contradiction.

Thus, two of the lines L_i are parallel but no third line is in the same parallel class as these two. Let $e = 0$. Since the point 0 of Σ is on L_4 and no three of the lines L_i are parallel, the set J_0 contains at most 7 non-zero points of Σ. Thus, there is a non-zero $c(0)$ in F such that $U[c(0), 0]$ does not contain any of the points R_i. By Lemma 6.4, $U[c(0), 0]$ has disjoint with $\mathcal{B}$, a contradiction.

Q.E.D.

LEMMA 7.4. *Given any line g of Π, $|g \cap \mathcal{B}| \leq 2$.*

PROOF: Assume, by way of contradiction, that $|g \cap \mathcal{B}| \geq 3$. By Lemma 7.3, $|g \cap \mathcal{B}| = 3$. Label the points in $\mathcal{B} \setminus g$ as R_i, $i = 1,\ldots,7$. Also by Lemma 7.3, there are $\binom{7}{2} = 21$ distinct lines R_iR_j, $i \neq j$. Thus, since $21 > 2 \times 10$, there is a point $Q \in g$ such that Q lies on 3 of these lines R_iR_j. Without loss of generality assume that R_2R_3, R_4R_5 and R_6R_7 are these lines. Again by Lemma 7.3, there is an integer $2 \leq i \leq 5$ so that R_1R_i intersects g and R_6R_7 at points P and T, respectively, not in $\mathcal{B}$. Without loss of generality assume $i = 2$. Coordinatize Π so that $P=(1,0,0)$, $Q=(0,1,0)$, $T=(0,0,1)$, $R_5=(1,1,1)$ and $R_i=(r_i, s_i, 1)$ for $i = 1,\ldots,4$. Then $g=(0,0,1)^t$ and $l := R_6R_7 = (1,0,0)^t$. Note that $s_1 = 0$, $s_2 = 0$, $r_2 = r_3$, $s_3 \neq 0$, $r_4 = 1$ and $s_4 \neq 0$. Set $r = r_2$. For any given e in F, define $L_i := L[r_i, f(e, s_i)]$ for $i = 1,\ldots,4$, $L_5 := L[1, f(e,1)]$ and $J_e = L_1 \cup \ldots \cup L_5$. By Lemma 6.2, L_4 and L_5 are parallel, and L_2 and L_3 are parallel.

Assume that $r \in K$. Then by Lemma 6.2, the lines L_2, L_3, L_4 and L_5 are in the same parallel class Δ. By Lemma 6.3, for the 5 values of e not equal to 0, $2s_3^3$, $2s_4^3$ or 2, the lines L_2, L_3, L_4 and L_5 do not contain the point 0 of Σ and, hence, constitute at most 2 lines of Δ. For these 5 values of e, the line L_1 also does not contain the point 0 by Lemma 6.3. Thus, for each of these 5 values of e, the set J_e contains at most 7 points of Σ. Hence, for each of these 5 values of e, there is a

non-zero $c(e)$ in F so that the unital $U[c(e), e]$ does not contain the points $R_1,...,R_5$. Since $|l \cap \mathcal{B}| < 5$, Lemma 6.4 implies the existence of a unital $U[c(e), e]$ which has disjoint intersection with $\mathcal{B}$, a contradiction. Thus, $r \notin K$.

Assume $r_1 \notin K$. As is easily checked, the points σ^2 and σ^6 of the affine plane Σ are on the line $L[1, 0]$. (Recall that σ is a primitive root of F as defined at the beginning of this chapter.) Let $e = 0$. Since r and r_1 are not in K, Lemma 6.2 implies that neither L_1 nor L_2 contains the point σ^2 or σ^6. Since $s_4 \neq 0$, neither does L_4 nor L_5. Since $r \notin K$, L_3 can not contain both σ^2 and σ^6. Thus, the set J_0 does not contain the 8 non-zero points of Σ. Hence, there is a non-zero $c(0)$ in F so that the unital $U[c(0), 0]$ has disjoint intersection with $\mathcal{B}$, a contradiction. So, $r_1 \in K$, and since $r_1 \neq 0$ or 1, must equal 2. This implies that L_1, L_4 and L_5 are in the same parallel class Δ. If $s_4^4 = 1$, then L_1, L_4 and L_5 constitute exactly 2 lines of Δ and the point 0 is on L_1. Since L_3 is not in Δ and the point 0 is on L_2, the set J_0 contains at most 7 non-zero elements of F. As above, this implies the existence of a unital $U[c(0), 0]$ which has disjoint intersection with $\mathcal{B}$. Hence, $s_4^4 = 2$.

To summarize: $r_1 \in K$, $r \notin K$, $s_4^4 = 2$, the 3 lines L_1, L_4 and L_5 are in the same parallel class Δ and the 2 lines L_2 and L_3 are in a parallel class Γ distinct from Δ.

We now want to show that there exists 3 values of e such that for each value the set J_e does not contain the 8 non-zero points of Σ. This would imply that for these 3 values of e there exists three distinct non-zero values $c(e)$ in F such that the unitals $U[c(e), e]$ do not contain the points $R_1,...,R_5$. As $l \backslash \{Q\}$ contains at most 2 points of $\mathcal{B}$, Lemma 6.4 would then imply the existence of a unital $U[c(e), e]$ which has disjoint intersection with $\mathcal{B}$, a contradiction.

First, we want to prove the existence of 7 values of e such that the lines L_1, L_4 and L_5 constitute at most 2 lines of Δ. For each of the 6 values of e not equal to $2s_4^3$, 2 or 0, none of these 3 lines contain the point 0 of the affine plane Σ. Hence,

for these 6 values of e, the 3 lines constitute at most 2 lines of Δ. If $e = 2$, then $2e^4 = (e + s_4^3)^4$ and L_1=L_4 unless $(2 + s_4^3)^4 = 2$. If $(2 + s_4^3)^4 = 2$, then for $e = 2s_4^3$, since $s_4^4 = 2$, one has that $2e^4 = 1$. So, $(e + 1)^4 = (2s_4^3 + 1)^4 = (s_4^3 + 2)^4 = 1$ and, hence, L_1=L_5. Thus, there are 7 values of e so that L_1, L_4 and L_5 constitute at most 2 lines of Δ. Let H denote the set of these 7 distinct values of e.

Second, we want to prove the existence of 5 values of e so that the lines L_2 and L_3 are either equal or one of them contains the point 0 of Σ. Clearly, L_2 contains the point 0 if $e = 0$ and L_3 contains the point 0 if $e = 2s_3^3$. Since $e^4 = (e + s_3^3)^4$ for values of e equal to s_3^3, σs_3^3 or $\sigma^3 s_3^3$, the lines L_2 and L_3 are equal for these 3 distinct values of e. Let G denote the set of these 5 distinct values of e.

The set H∩G contains at least 3 values. For each of these values of e, the set J_e contains at most 7 non-zero points of Σ. Thus, we have proved Lemma 7.4.

Q.E.D.

Therefore, the 10 points of $\mathcal{B}$ constitute an oval of Π=PG(2, 9). The proof of the following lemma will therefore complete the proof of Theorem 7.1.

LEMMA 7.5. *For q an odd prime power, an oval Θ of Ψ=PG(2, q^2) is not a blocking set of H(q).*

PROOF: As is well known, an oval in Ψ can be represented as a conic with coefficients from F=GF(q^2). [17, Theorem 1] It is also well known that all conics in Ψ are projectively equivalent. [12, Theorem 2.36]

Let w be a primitive root of K=GF(q) and define F as the polynomial ring K[T] modulo the ideal generated by $T^2 - w$. Let $\sigma = aT^2 + b$ be a primitive root of F, a and b in K.

Case 1. Assume q is congruent to 3 modulo 4.

Since all conics are projectively equivalent, represent Θ by the equation $X^2 + Y^2 - \sigma Z^2 = 0$; that is, a point P=$(x,y,z)$ from Ψ is also in Θ if and only if $x^2+y^2-\sigma z^2 = 0$. Let U be the Hermitian unital represented by the equation $X^{q+1}+Y^{q+1}+bZ^{q+1} = 0$; that is, point P=$(x,y,z)$ is in U if and only if $x^{q+1}+y^{q+1}+bz^{q+1} = 0$.

Assume that point P=(x,y,z) is in $\Theta\cap$U. If $z=0$, then $x^2 = -1$ and $x^{q+1} = -1$. But, since q is congruent to 3 modulo 4, $x^2 = -1$ implies $x^{q+1} = (x^2)^{\frac{q+1}{2}} = 1$. So, $z = 1$. If $y = 0$, then $x^2 = \sigma$ and $x^{q+1} = -b$ imply $\sigma^{\frac{q^2-1}{2}} = 1$, contrary to σ being primitive. So, $y \neq 0$. Similarly, $x \neq 0$.

Set $x = cT+d$ and $y = eT+f$, where c, d, e and f are in K. Then $x^2+y^2 = \sigma$ implies $d^2+f^2+(c^2+e^2)w = b$, and $x^{q+1}+y^{q+1} = -b$ implies $d^2+f^2-(c^2+e^2)w = -b$. Adding one gets $d^2 + f^2 = 0$. As q is congruent to 3 modulo 4, $d = 0 = f$. Thus, $x^2+y^2 = \sigma$ is an element of K, a contradiction.

Case 2. Assume q is congruent to 1 modulo 4.

Represent Θ by $X^2 - Y^2 - \sigma Z^2 = 0$ and U by $X^{q+1} + Y^{q+1} - bZ^{q+1} = 0$. Suppose that the point $P = (x,y,z)$ belongs to the set $\Theta\cap$U. If $z = 0$, then $x^2 = 1$ and $x^{q+1} = -1$, clearly impossible. So, $z \neq 0$. If $y = 0$, then $x^2 = \sigma$ and $x^{q+1} = b$ imply $\sigma^{\frac{q^2-1}{2}} = 1$, contrary to σ being primitive. Thus, $y \neq 0$. Similarly, $x \neq 0$. As above, set $x = cT+d$ and $y = eT+f$. Then $x^2-y^2 = \sigma$ implies $d^2-f^2+(c^2-e^2)w = b$, and $x^{q+1}+y^{q+1} = b$ implies $d^2+f^2-(c^2+e^2)w = b$. Subtracting one gets that $w = (\frac{f}{c})^2$ is a square in K, a contradiction, unless $c = 0 = f$. If $c = 0 = f$, then $x^2-y^2 = \sigma$ is an element of K, another contradiction.

Q.E.D.

To summarize, in Chapters 6 and 7 it has been shown that $\mathcal{C}(H(3))$ =PG(2,9) and $\mathcal{C}(H(q)) \neq$PG(2,q^2) for $q \geq 23$.

CHAPTER 8

FINAL REMARKS

As this dissertation does not lend itself to a conclusion or comprehensive summary, let the author end it by commenting upon possible future research related to the work done here.

Because Theorem 4.1 is true for $t \geq 9$ and $1 \leq e \leq 2t-2$, it would be desirable to have a theorem similar to Theorem 5.1 but with these weaker restrictions on t and e. The author believes that such a theorem does exist and is at present trying to show it.

A. Blokhuis and A. E. Brouwer have shown that if q is odd, greater than 7 and not 27, then any blocking set of PG(2,q) has cardinality at least $q + (2q)^{\frac{1}{2}} + 1$. [5] Is there such a result for all projective planes? As a first step, one might consider blocking sets of Rédei type.

In Chapter 6 the large difference between the lower and upper bounds given on the cardinality of a committee of $H(q)$ is unpleasant, but it appears that closing this gap is difficult. The author and others have worked on it with no success. The author has also tried to find something significant to say concerning the size of a committee in $H(5)$, but also with no success.

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BIOGRAPHICAL SKETCH

Cyrus Kitto was born December 28, 1947, in Los Angeles, California. He received an undergraduate degree in liberal arts from Rollins College in 1970. He received a master's degree in mathematics from the University of Florida in 1987.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

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